ON THE REGULARITY OF A MODEL NON-NEWTONIAN FLUID

by

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Abstract

Existence and regularity of steady and unsteady solutions of a PDE describing the motion of a prototypical incompressible fluid with shear dependent viscosity are studied. The regularity theory is approached by studying the associated elliptic operator. A summary of the classical technique of difference quotients applied to non-linear elliptic systems is given by applying it to the elliptic system associated with a vector Burgers-like system. Interior regularity is proved for a general class of Stokes-like elliptic operators using a new solenoidal test function that permits the application difference quotient methods to systems with a divergence free constraint. Existence for steady solutions of the incompressible fluid PDE is proven; interior regularity follows immediately from regularity of the Stokes-like elliptic system. Existence and interior regularity for time dependent solutions are proven.

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Chapter 1
Introduction

In this thesis we study an incompressible fluid with equation of motion

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla \pi + 2\,\mathrm{div}((\nu_1 + \nu_2 |\mathbf{D}\mathbf{u}|^2)\mathbf{D}\mathbf{u}) + \mathbf{f} \\ \mathrm{div}\,\mathbf{u} &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0 \end{aligned} \tag{1.1}$$

where $\mathbf{D}\mathbf{u}$ is the symmetric part of the gradient of $\mathbf{u}$. For simplicity, we will work on a bounded domain Ω contained in $\mathbf{R}^3$. Our primary goal is to investigate the regularity of solutions of this equation, and to do this we focus on the regularity of the associated nonlinear Stokes-like system

$$\begin{aligned} -2\,\mathrm{div}((\nu_1 + \nu_2 |\mathbf{D}\mathbf{u}|^2)\mathbf{D}\mathbf{u}) &= -\nabla \pi + \mathbf{f} \\ \mathrm{div}\,\mathbf{u} &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{1.2}$$

We are motivated to study (1.1) for several reasons. Of course, this is a physical model and exhibits so-called "shear thickening" effects seen in some non-Newtonian fluids. However, this is only a specific case of a general class of fluids with power law type Cauchy stress tensors

$$T_{ij} = 2(\nu_1 + \nu_2 |\mathbf{D}\mathbf{u}|^q) D_{ij}\mathbf{u}$$

which are themselves a specific case of models with stress tensors satisfying Stoke's hypothesis on the stress-strain relationship,

$$T_{ij} = f_1(|\mathbf{Du}|, \det \mathbf{Du}) D_{ij}\mathbf{u} + f_2(|\mathbf{Du}|, \det \mathbf{Du}) \left[\mathbf{Du}^2\right]_{ij}.$$

On the other hand, the specific model is associated with the Smagorinsky turbulence model [Sma63]. For a uniform mesh with mesh size h this model can be written with stress tensor

$$2\nu_1(1 + h^2|\mathbf{Du}|^2)D_{ij}\mathbf{u}$$

which is obviously related to the model studied in this thesis. Most importantly, the reason for addressing a prototypical problem is that we can focus on the real difficulties of the regularity theory without worrying about being bogged down in details required for generalization. Since we will not be able to come up with a complete regularity theory in the end, it does not appear to be a large loss to use a specific model with a structure that is particularly amenable to analysis. Indeed, the nagging open questions of regularity for nonlinear elliptic systems are especially clear and troubling for the model (1.2). Since the viscosity of this fluid is $(\nu_1 + \nu_2|\mathbf{Du}|^2)$, we see that the viscosity increases wherever $\mathbf{Du}$ does, and we would expect that the larger viscosity would then act to damp $\mathbf{Du}$ at those places where it is large. Thus we can imagine this as some sort of self-governing fluid and we would expect solutions to exhibit at least the regularity properties of solutions of the Navier-Stokes equations. In particular, in smooth domains with smooth forces we should be able to prove the existence of classical solutions, at least on some time interval $(0, T)$. The fact of the matter is that regularity theory for nonlinear elliptic and parabolic systems (and so also this thesis) is currently unable to address this question, even without the added complications of the solenoidal constraint considered here. With classical solutions as an end-goal, it seems unnecessary to focus too heavily on generalizations when we cannot even obtain desired answers for the specific case.

Our hope is that the arguments used in this thesis would be accessible to a reader with a basic functional analysis background as well as some familiarity Sobolev spaces and the spaces of divergence free functions such as can be found in a cursory reading of Chapters II and III of [Gal94]. Whenever possible we have chosen to use a simple argument rather than a more complicated, though perhaps more powerful, one particularly when the underlying principle is the same. For example, in addressing the existence of solutions of (1.2) we have chosen to use the easily proven fact that if $\mathbf{u}^k$ converges weakly to $\mathbf{u}$, then $\|\mathbf{u}\| \leq \lim_k \|\mathbf{u}^k\|$ rather than the more general principle of weak lower semicontinuity of convex functionals.

The structure of the thesis is as follows. In Chapter 2 we prove the existence and uniqueness of weak solutions of (1.2). We introduce the pressure in this Chapter and point out the surprising features of its apparent irregularity. From this we move on in Chapter 3 to a review of how the classical technique of difference quotients can be applied to the study of the related non-linear "vector-Burgers-like" system

$$\begin{aligned} -\operatorname{div}((\nu_1 + \nu_2 |\nabla \mathbf{u}|^2) \nabla \mathbf{u}) &= \mathbf{f} \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{1.3}$$

We do this for two reasons. First, it provides the reader with an introduction to the application of difference quotient techniques to non-linear systems, particularly those with growth condition different from the Laplacian and Stokes systems (i.e. growth condition (1.5) below with $p > 2$). More importantly, it illustrates why these techniques cannot be applied to systems with solenoidal condition directly and it therefore provides context for the new results in Chapter 4 which form the center-piece of the thesis. Building on the intuition developed in Chapter 3, we prove in Chapter 4 our fundamental regularity result for (1.2). We show that if $\mathbf{f}$ is in $L^2(\Omega)$ then the weak solution of (1.2) has second derivatives locally in L^2. Actually, we prove more than this. If $\mathbf{T}$ is a C^1 function mapping $\mathbf{R}^{n\times n}_{symm}$ to $\mathbf{R}^{n\times n}_{symm}$ such that for some $p \geq 2$

$$\partial_{kl} T_{ij}(\mathbf{A}) B_{ij} B_{kl} \geq c_1 (1 + |\mathbf{A}|^{p-2}) |\mathbf{B}|^2 \tag{1.4}$$

$$|\partial_{kl} T_{ij}(\mathbf{A})| \leq c_2 (1 + |\mathbf{A}|^{p-2}) \tag{1.5}$$

for all symmetric n dimensional matrices $\mathbf{A}$ and $\mathbf{B}$, we prove that weak solutions of the system

$$-\operatorname{div} \mathbf{T}(\mathbf{Du}) = -\nabla \pi + \mathbf{f}$$

$$\operatorname{div} \mathbf{u} = 0$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

have second derivatives locally in L^2. To do this we introduce a new test function that allows difference quotient techniques to be extended to system (1.2) and related systems. We have chosen to violate in this Chapter our avoidance of generality for a couple of reasons. Firstly, the arguments to get the general proof are no different from those for the specific case, save for some preliminary lemmas that are well motivated by our work in Chapter 3. So we do little extra work to get the stronger result. Also, the conditions (1.4) and (1.5) are easily seen to be analogous to those of ellipticity and growth standard in the theory of elliptic equations and systems. Thus we see that the result is a genuine extension of the theory of difference quotients to non-linear elliptic systems with solenoidal constraint. When presented in this context, not only is the technique new, but it also provides an extension of the class of stress tensors $\mathbf{T}$ for which we have regularity results. Therefore it seems appropriate to exhibit the stronger theorem. In Chapter 5 we apply the interior regularity result of Chapter 4 to steady and unsteady solutions of 1.1. Let Ω' be any open subset of Ω with closure contained in Ω. In the steady case, we prove the existence of solutions and show that every steady solution with $\mathbf{f}$ in $L^2(\Omega)$ has second derivatives in $L^2(\Omega')$. In the unsteady case we show that if the initial velocity is solenoidal and in $L^2(\Omega)$ and if the forcing term $\mathbf{f}$ is in $L^2(\Omega \times [0,T])$ then there exists a unique solution $\mathbf{u}$ with second spatial derivatives in $L^2(\Omega' \times [0,T])$ and that $t\mathbf{u}$ has a time derivative in $L^2(\Omega \times [0,T])$. The proof avoids the technical arguments required for existence proofs of weak solutions. Instead, we are motivated by the existence proof for Navier-Stokes equations [Hey80] wherein the existence of regular solutions is proven directly rather than proving weak solutions exist and then examining their regularity. In the end, though, we do not get particularly regular solutions for (1.1) (and certainly not classical ones) and so we are only motivated by some of the preliminary ideas of [Hey80].

The regularity of the motion of an incompressible fluid evolving according to (1.1) has been studied since the pioneering work of Ladyzhenskaya. For a class of fluids including those modeled by (1.1) she showed the unique existence of weak solutions, in contrast to the Navier-Stokes equations for which such a result is not yet known in three dimensions. In the steady case, one can find in the work of Giaquinta and Modica [GM82] results concerning the almost-everywhere Hölder continuity of the gradients of solutions of a class of equations similar to (1.2), except these equations have growth property of the type (1.5) with $p = 2$ only and therefore do not exhibit the difficulties associated with the pressure we shall see later. There is a technique in [MNRR96] for the unsteady case that when applied to the Stokes-like system seems likely to give $W^{2,2}_{loc}$ solutions as well as some form of boundary regularity. However, these calculations have not yet been done and would apply directly only in the case that $\mathbf{T}$ were derived from a potential, by which we mean $T_{ij}(\mathbf{Du}) = \partial_{ij} F(|\mathbf{Du}|^2)$ for some scalar function F. For the unsteady problem we have recent work [MNR93] [BBN94] [MNRR96] in the case of periodic boundary conditions wherein the global existence of regular solutions with second derivatives in $L^2(\Omega \times [0,T])$ are proven for a class of equations including (1.1). For Dirichlet boundary conditions, new difficulties appear, as seen in Chapter 4, and the known results are more sketchy. We have an existence proof of classical solutions for small data from Amann [Ama94]. For large data, there is an existence proof in [MNR96] of regular solutions, i.e. $L^2(0,T;W^{2,4/3}(\Omega))$. The techniques of our proof are different from those in [MNR96], so we obtain a different perspective on the solutions. For example, our proof makes explicit the property that the second derivatives are in in $L^2(0,T;W^{2,2}(\Omega'))$ for open sets Ω' with closure contained in Ω. We have then made apparent the troubling question of whether singularities of type $L^{\frac{4}{3}}$ occur in the second derivatives at the boundary. Also, when take in the context of a generalized stress tensor $\mathbf{T}$, our existence proof extends in some respects the results of [MNR96]. The extension of the results of Chapter 4 to the generalized case is taken up in [Max97]. We should ephasize here, though, that since we do not obtain boundary regularity we do not fully recover the results of [MNR96].

1.1 Notation

We end our introductory remarks by outlining our notation. We will use ∇ to denote the gradient operator and $\mathbf{D}$ the symmetric part of the gradient. That is, $\mathbf{Du}$ has components $(\mathbf{Du})_{ij} = D_{ij}\mathbf{u} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. If f is a function defined on square matrices, we will write $\partial_{ij} f$ to mean the derivative with respect to the i^{th},j^{th} component of its argument.

Let $\{\mathbf{e}_k\}_{k=1}^n$ denote the standard basis of $\mathbf{R}^n$. For any function f defined on $\mathbf{R}^n$ we define $\tau_{h,s} f(\mathbf{x})$ to be the difference quotient $\frac{f(\mathbf{x}+h\mathbf{e}_s)-f(\mathbf{x})}{h}$. In addition to difference quotients, we will have occasion to use a smearing operator $\sigma_{h,s}$ defined by $\sigma_{h,s} f(\mathbf{x}) = \int_0^1 f(\mathbf{x} + th\mathbf{e}_s)\,dt$. In all that follows we will use the usual Einstein summation notation with the exception that indices appearing in the operators τ and σ should not be summed unless this is written explicitly. Therefore, $\tau_{-h,j}\tau_{h,j}u$ would not denote a difference quotient “Laplacian”, but $\sum_j \tau_{-h,j}\tau_{h,j}u$ would.

If X is a Banach space, we will use $|\cdot|$ to denote its norm if it is finite dimensional, $||\cdot||$

if it is L^2, $||\cdot||_p$ if it is L^p and $||\cdot||_X$ otherwise. Also, we will use the generic notation X^* to mean the dual of X. We use $\langle f, v\rangle$ to denote the value of the functional f taken at v. In the case of L^2, we use $(\cdot,\cdot)$ to denote the inner product.

We denote by $W^{n,p}(\Omega)$ the usual Sobolev space with $W_0^{n,p}(\Omega)$ the subspace generated by taking the closure in $W^{n,p}(\Omega)$ of $C_0^\infty(\Omega)$, the set of smooth functions with compact support. Let $\mathcal{J}(\Omega)$ denote the subset of $C_0^\infty(\Omega)$ that are also divergence free (we will not make distinction between spaces of scalar and vector valued functions as this will always be clear from the context). Let $J_0^{n,p}(\Omega)$ denote the closure of $\mathcal{J}(\Omega)$ in $W_0^{n,p}(\Omega)$. [1] We will simply write J^p for $J_0^{0,p}$ and J for J^2. While our notation for divergence free spaces is not traditional, it is hopefully both consistent and clear.

If Ω is an open subset of $\mathbf{R}^n$ and $0 < \delta < T$ we write Ω_T and $\Omega_{[\delta,T]}$ to denote the time cylinders $\Omega\times[0,T]$ and $\Omega\times[\delta,T]$ respectively. We use the standard notation for Bochner spaces. If $f \in L^p(0,T;X)$ then $\int_0^T ||f||_X^p\,dt < \infty$. We identify $L^p(0,T;L^p(\Omega))$ with $L^p(\Omega_T)$.

[1]Since we will always work on a bounded domain in this document, we need not worry about the troubles that arise from this definition in unbounded domains.

Chapter 2
Weak Solutions of the Stokes-like System

2.1 Existence and Uniqueness

In this section, we shall concern ourselves with the nonlinear elliptic system of equations defined on Ω, a bounded open set in $\mathbf{R}^n$, namely

$$\begin{aligned} -2\partial_j \left(\left(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2\right) D_{ij}\mathbf{u}\right) &= \partial_i \pi + f_i \\ \partial_i u_i &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{2.1}$$

If $\mathbf{f}$ is in $\left(J_0^{1,4}\right)^*$, we define a *weak* solution of (2.1) to be a function $\mathbf{u}$ in $J_0^{1,4}$ such that

$$\int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u}\partial_j\phi_i = \langle \mathbf{f}, \phi\rangle \tag{2.2}$$

for all solenoidal ϕ in $C_0^\infty(\Omega)$. It is clear by simple integration by parts that any sufficiently smooth solution of (2.1) is also a weak solution. Moreover, for any fixed $\mathbf{u} \in J_0^{1,4}$ we obtain from Hölder's inequality that for every ϕ in $\mathcal{J}(\Omega)$,

$$\begin{aligned} \left| \int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u}\partial_i\phi_j \, d\mathbf{x} - \langle \mathbf{f}, \phi\rangle \right| &\le 2\nu_1 \left|\Omega\right|^{\frac{1}{2}} ||\mathbf{Du}||_{L^4} ||\phi||_{J_0^{1,4}} + \\ &\quad + 2\nu_2 ||\mathbf{Du}||_{L^4}^3 ||\phi||_{J_0^{1,4}} + ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} ||\phi||_{J_0^{1,4}} \\ &\le \Big(2\nu_1 \left|\Omega\right|^{\frac{1}{2}} ||\mathbf{Du}||_{L^4} + 2\nu_2 ||\mathbf{Du}||_{L^4}^3 + \\ &\qquad + ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} \Big) ||\phi||_{J_0^{1,4}}. \end{aligned}$$

Since solenoidal C_0^∞ functions are dense in $J_0^{1,4}$, it follows that for fixed $\mathbf{u}$ the integral in (2.2) defines a continuous linear functional on $\phi \in J_0^{1,4}$. In particular, if $\mathbf{u}$ is a weak solution, then

$$\int_\Omega 2(\nu_1 + \nu_2 |\mathbf{Du}|^2) D_{ij}\mathbf{u}\partial_j v_i \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle = 0 \tag{2.3}$$

for all $\mathbf{v} \in J_0^{1,4}$.

To investigate the solubility of (2.2) we will consider the functional on $J_0^{1,4}$ defined by

$$\mathcal{F}(\mathbf{u}) = \int_\Omega \nu_1 |\mathbf{Du}|^2 + \frac{\nu_2}{2} |\mathbf{Du}|^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u} \rangle \tag{2.4}$$

That $\mathcal{F}$ is well defined on $J_0^{1,4}$ is clear since

$$\begin{aligned}
|\mathcal{F}(\mathbf{u})| &\leq \int_\Omega \nu_1 |\mathbf{Du}|^2 + \frac{\nu_2}{2} |\mathbf{Du}|^4 \, d\mathbf{x} + |\langle \mathbf{f}, \mathbf{u} \rangle| \\
&\leq \int_\Omega \nu_1 |\nabla \mathbf{u}|^2 + \frac{\nu_2}{2} |\nabla \mathbf{u}|^4 \, d\mathbf{x} + ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} ||\mathbf{u}||_{J_0^{1,4}} \\
&\leq \nu_1 |\Omega|^{\frac{1}{2}} ||\mathbf{u}||^2_{J_0^{1,4}} + \frac{\nu_2}{2} ||\mathbf{u}||^4_{J_0^{1,4}} + ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} ||\mathbf{u}||_{J_0^{1,4}}.
\end{aligned}$$

We now apply the theory of variational integrals to the functional $\mathcal{F}$ to obtain the existence of a minimizer for $\mathcal{F}$. Since critical points of $\mathcal{F}$ are shown to be solutions of (2.2) we will have in hand the existence of a weak solution. Moreover, by using the simple form of $\mathcal{F}$, we will be able to prove these things without calling directly upon the standard results concerning weak upper semi-continuous functionals.

As is usual, we define a critical point of $\mathcal{F}$ to be a function $\mathbf{u}$ in $J_0^{1,4}$ such that for any $\mathbf{v}$ in $J_0^{1,4}$, $\frac{d}{ds}\mathcal{F}(\mathbf{u} + s\mathbf{v})\big|_{s=0} = 0$. Since $\mathcal{F}(\mathbf{u} + s\mathbf{v})$ is just a polynomial in s, we can easily compute the derivative of $\mathcal{F}(\mathbf{u} + s\mathbf{v})$ with respect to s and evaluate it at $s = 0$. Doing this yields

$$\begin{aligned}
\frac{d}{ds}\mathcal{F}(\mathbf{u} + s\mathbf{v})\bigg|_{s=0} &= \int_\Omega 2(\nu_1 + \nu_2 |\mathbf{Du}|^2) D_{ij}\mathbf{u} D_{ij}\mathbf{v} \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle \\
&= \int_\Omega 2(\nu_1 + \nu_2 |\mathbf{Du}|^2) D_{ij}\mathbf{u}\partial_i v_j \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle
\end{aligned} \tag{2.5}$$

where we arrived at the last line using the fact that if $\mathbf{A}$ is a symmetric matrix and $\mathbf{B}$ is an anti-symmetric matrix, $A_{ij}B_{ij} = 0$. It its clear from (2.5) that a critical point is weak solution.

Suppose $\mathcal{F}$ has a minimizer $\mathbf{u}$. We now show that $\mathbf{u}$ is a critical point of $\mathcal{F}$, and therefore a weak solution. If $\mathbf{v}$ is any other element of $J_0^{1,4}$, then $\mathcal{F}(\mathbf{u} + s\mathbf{v})$ is a fourth order polynomial in s; call it $r(s)$. Since $\mathbf{u}$ is a minimizer, $r(0) \leq r(s)$ for all s. Thus, $r'(0) = 0$, which can be written as $\frac{d}{ds}\mathcal{F}(\mathbf{u} + s\mathbf{v})\big|_{s=0} = 0$. Since $\mathbf{v}$ is arbitrary, $\mathbf{u}$ is a critical point.

Given what we have just seen, to show existence of a weak solution it would be sufficient to show existence of a minimizer. This is what we turn to now.

First, we will show that $\mathcal{F}$ is coercive and bounded below. To do this, we split the domain of $\mathcal{F}$ into two regions, one wherein the term $\langle \mathbf{f}, \mathbf{u} \rangle$ is dominant and the remainder of the space where this term in subordinate. We will need to use a case of the Korn inequality proved, for example, in [Neč66]: for $\mathbf{u} \in J_0^{1,4}$, there is a constant $k(\Omega) > 0$ such that $||\mathbf{u}||_{J_0^{1,4}} \leq k(\Omega)||\mathbf{Du}||_{L^4}$.

Let us now consider the case where $\langle \mathbf{f}, \mathbf{u} \rangle$ is subordinate. Suppose

$$||\mathbf{u}||^3_{J_0^{1,4}} \geq \frac{k(\Omega)}{\nu_2}\left(||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} + \gamma\right)$$

where $\gamma \geq 0$ will be determined later. Then

$$\begin{aligned}
\mathcal{F}(\mathbf{u}) &= \int_\Omega \nu_1 \left|\mathbf{Du}\right|^2 + \frac{\nu_2}{2}\left|\mathbf{Du}\right|^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u} \rangle \, d\mathbf{x} \\
&\geq \frac{\nu_2}{2}||\mathbf{Du}||^4_{L^4} - ||\mathbf{u}||_{J_0^{1,4}}||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} \\
&\geq \frac{\nu_2}{2k(\Omega)^4}||\mathbf{u}||^4_{J_0^{1,4}} - ||\mathbf{u}||_{J_0^{1,4}}||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} \\
&\geq \left(\frac{\nu_2}{2k(\Omega)^4}||\mathbf{u}||^3_{J_0^{1,4}} - ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*}\right)||\mathbf{u}||_{J_0^{1,4}} \\
&\geq \gamma||\mathbf{u}||_{J_0^{1,4}}
\end{aligned}$$

Taking $\gamma = 1$ gives us the desired coercivity. Taking $\gamma = 0$ we have shown that $\mathcal{F}$ is non-negative when $||\mathbf{u}||^3_{J_0^{1,4}} \geq \frac{2k(\Omega)^4}{\nu_2}||\mathbf{f}||_{\left(J_0^{1,4}\right)^*}$. Now we find a lower bound when $||\mathbf{u}||^3_{J_0^{1,4}} < \frac{2k(\Omega)^4}{\nu_2}||\mathbf{f}||_{\left(J_0^{1,4}\right)^*}$ by neglecting the positive terms in $\mathcal{F}$ to get

$$
\begin{aligned}
\mathcal{F}(\mathbf{u}) &= \int_\Omega \nu_1 \left|\mathbf{Du}\right|^2 + \frac{\nu_2}{2}\left|\mathbf{Du}\right|^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u}\rangle \\
&\geq - \langle \mathbf{f}, \mathbf{u}\rangle \\
&\geq -||\mathbf{u}||_{J_0^{1,4}}||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} \\
&\geq -\left(\frac{2k(\Omega)^4}{\nu_2}\right)^{\frac{1}{3}}||\mathbf{f}||^{\frac{4}{3}}_{\left(J_0^{1,4}\right)^*}
\end{aligned}
$$

Thus we have obtained that $\mathcal{F}$ is bounded below by $-\left(\frac{2k(\Omega)^4}{\nu_2}\right)^{\frac{1}{3}}||\mathbf{f}||^{\frac{4}{3}}_{\left(J_0^{1,4}\right)^*}$.

We are now able to find a minimizer for $\mathcal{F}$. Since $\mathcal{F}$ is bounded below, there exists a sequence $\mathbf{u}^n$ of terms in $J_0^{1,4}$ such that $\mathcal{F}(\mathbf{u}^n) \to \inf_{\mathbf{v}\in J_0^{1,4}}\mathcal{F}(\mathbf{v})$. By coercivity, the sequence is bounded in $J_0^{1,4}$ and so converges weakly in $J_0^{1,4}$ to some $\mathbf{u}$. We now show that $\mathcal{F}(\mathbf{u})$ is the minimal value for $\mathcal{F}$.

Recall from elementary functional analysis that if $\mathbf{u}^n$ converges weakly to $\mathbf{u}$ in a Banach space B, then $\liminf_{n\to\infty}||\mathbf{u}^n||_B \geq ||\mathbf{u}||_B$. Since $||\mathbf{Du}||_{L^4} \leq ||\mathbf{u}||_{J_0^{1,4}} \leq k(\Omega)||\mathbf{Du}||_{L^4}$, as cited before, we can take $||D\cdot||_{L^4}$ as the norm on $J_0^{1,4}$. Since $\{\mathbf{u}^n\}$ converges weakly to $\mathbf{u}$ in $J_0^{1,4}$ we have $\liminf_{n\to\infty}||\mathbf{Du}^n||_{L^4} \geq ||\mathbf{Du}||_{L^4}$. Also, since Ω is bounded and $\mathbf{u}^n$ converges weakly to $\mathbf{u}$ in $J_0^{1,4}$, it also converges weakly to $\mathbf{u}$ in $J_0^{1,2}$. We can easily see by a couple of integrations by parts that $||\mathbf{Du}||_{L^2} = ||\mathbf{u}||_{J_0^{1,2}}$. So, $\liminf_{n\to\infty}||\mathbf{Du}^n||_{L^2} \geq ||\mathbf{Du}||_{L^2}$. Using these fact together with the weak continuity of linear functionals it follows that

$$
\begin{aligned}
\inf_{\mathbf{v}\in J_0^{1,4}} \mathcal{F}(\mathbf{v}) &= \lim_{n\to\infty} \mathcal{F}(\mathbf{u}^n) \\
&= \lim_{n\to\infty} \int_\Omega \nu_1 \left|\mathbf{D}\mathbf{u}^n\right|^2 + \frac{\nu_2}{2} \left|\mathbf{D}\mathbf{u}^n\right|^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u}^n \rangle \\
&= \lim_{n\to\infty} \int_\Omega \nu_1 \left|\mathbf{D}\mathbf{u}^n\right|^2 + \frac{\nu_2}{2} \left|\mathbf{D}\mathbf{u}^n\right|^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u} \rangle \\
&\geq \nu_1 \liminf_{n\to\infty} ||\mathbf{D}\mathbf{u}^n||_{L^2}^2 + \frac{\nu_2}{2} \liminf_{n\to\infty} ||\mathbf{D}\mathbf{u}^n||_{L^4}^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u} \rangle \\
&\geq \nu_1 ||\mathbf{D}\mathbf{u}||_{L^2}^2 + \frac{\nu_2}{2} ||\mathbf{D}\mathbf{u}||_{L^4}^4 \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{u} \rangle \\
&= \mathcal{F}(\mathbf{u}).
\end{aligned}
$$

Thus $\mathbf{u}$ is in fact a minimizer for our functional and a weak solution of (2.2).

The uniqueness of weak solutions follow from the convexity and smoothness of the functional $\mathcal{F}$. We can show this in an elementary way using finite dimensional arguments.

Suppose $\mathbf{u}^1$ and $\mathbf{u}^2$ are two distinct weak solutions of (2.2). Then $\mathcal{F}(s\mathbf{u}^1 + (1-s)\mathbf{u}^2)$ is a fourth order polynomial in s, call it $r(s)$, with derivatives

$$
\begin{aligned}
r'(s) = \int_\Omega & 2\nu_1 D_{ij}\left(s\mathbf{u}^1 + (1-s)\,\mathbf{u}^2\right) D_{ij}(\mathbf{u}^1 - \mathbf{u}^2) + \\
& + 2\nu_2 \left|\mathbf{D}\left(s\mathbf{u}^1 + (1-s)\,\mathbf{u}^2\right)\right|^2 D_{ij}\left(s\mathbf{u}^1 + (1-s)\,\mathbf{u}^2\right) D_{ij}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} - \\
& - \left\langle \mathbf{f}, \left(\mathbf{u}^1 - \mathbf{u}^2\right)\right\rangle
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
r''(s) = \int_\Omega & 2\nu_1 \left|\mathbf{D}\left(\mathbf{u}^1 - \mathbf{u}^2\right)\right|^2 + 4\nu_2 \left|D_{ij}\left(s\mathbf{u}^1 + (1-s)\,\mathbf{u}^2\right) D_{ij}(\mathbf{u}^1 - \mathbf{u}^2)\right|^2 + \\
& + 2\nu_2 \left|\mathbf{D}\left(s\mathbf{u}^1 + (1-s)\,\mathbf{u}^2\right)\right|^2 \left|\mathbf{D}(\mathbf{u}^1 - \mathbf{u}^2)\right|^2 \, d\mathbf{x} \\
> 0. &
\end{aligned} \tag{2.7}
$$

From equation (2.6) using (2.3) we obtain

$$\begin{aligned} r'(0) &= \int_\Omega 2\nu_1 D_{ij}u^2 D_{ij}(\mathbf{u}^1 - \mathbf{u}^2) + 2\nu_2 \left|\mathbf{Du}^2\right|^2 D_{ij}\mathbf{u}^2 D_{ij}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} - \left\langle \mathbf{f}, \left(\mathbf{u}^1 - \mathbf{u}^2\right)\right\rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} r'(1) &= \int_\Omega 2\nu_1 D_{ij}\mathbf{u}^1 D_{ij}(\mathbf{u}^1 - \mathbf{u}^2) + 2\nu_2 \left|\mathbf{Du}^1\right|^2 D_{ij}\mathbf{u}^1 D_{ij}(\mathbf{u}^1 - \mathbf{u}^2)\, d\mathbf{x} - \left\langle \mathbf{f}, \left(\mathbf{u}^1 - \mathbf{u}^2\right)\right\rangle \\ &= 0. \end{aligned}$$

By the Mean Value Theorem, then, there is an s_0 in $(0,1)$ with $r''(s_0) = 0$. This contradicts the positivity of r'' as shown in (2.7). Thus there can be at most one weak solution. The results of this and the previous section can be summarized in the following theorem.

Theorem 2.1 *Let Ω be a bounded open subset of $\mathbf{R}^n$ and $\mathbf{f}$ be in $\left(J_0^{1,4}\right)^*(\Omega)$. Then there exists a unique $\mathbf{u} \in J_0^{1,4}$ such that:*

$$\int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u} D_{ij}\mathbf{v}\, d\mathbf{x} = \langle \mathbf{f}, \mathbf{v}\rangle$$

for all $\mathbf{v} \in J_0^{1,4}$. Moreover, there is a constant $c(\Omega)$ such that

$$||\mathbf{u}||_{J_0^{1,4}} \leq c(\Omega) \left(\frac{1}{\nu_2} ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*}\right)^{\frac{1}{3}}.$$

The final estimate of the theorem is easily derived:

$$\begin{aligned} \nu_2 2k(\Omega)||\mathbf{u}||^4_{J_0^{1,4}} &\leq 2\nu_2 ||\mathbf{Du}||^4_{L^4} \\ &\leq \int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u} D_{ij}u\, d\mathbf{x} \\ &= \int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u}\partial_i u_j\, d\mathbf{x} \\ &= \langle \mathbf{f}, \mathbf{u}\rangle \\ &\leq ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} ||\mathbf{u}||_{J_0^{1,4}}. \end{aligned}$$

2.2 Existence of a Pressure

Until now, we have not discussed the pressure which appears in (2.1) but not in the weak formulation. However, using standard results from Navier-Stokes theory we can show a reformulation of the weak formulation in which it appears.

Given any $\mathbf{f} \in \left(J_0^{1,4}\right)^*$ we have, since $J_0^{1,4}$ is a closed subspace of $W_0^{1,4}$ with the same norm, that

$$\begin{aligned} ||\mathbf{f}||_{\left(J_0^{1,4}\right)^*} &= \sup_{\mathbf{u} \in J_0^{1,4}} \frac{\langle \mathbf{f}, \mathbf{u} \rangle}{||\mathbf{u}||_{J_0^{1,4}}} \\ &= \sup_{\mathbf{u} \in J_0^{1,4}} \frac{\langle \mathbf{f}, \mathbf{u} \rangle}{||\mathbf{u}||_{W_0^{1,4}}}. \end{aligned}$$

By the Hahn-Banach theorem there exists $\tilde{\mathbf{f}} \in \left(W_0^{1,4}\right)^*$ such that its restriction to $J_0^{1,4}$ is just $\mathbf{f}$.

Let $\mathbf{u}$ be the weak solution of (2.2) for a given $\mathbf{f}$ and let $\tilde{\mathbf{f}}$ be an extension as described above. Then the linear functional $\mathcal{L}$ on $W_0^{1,4}$ defined by

$$\mathcal{L}(\mathbf{v}) = \int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u} \partial_j v_i \, d\mathbf{x} - \left\langle \tilde{\mathbf{f}}, \mathbf{v} \right\rangle$$

is continuous. From (2.2) we see that $\mathcal{L}(\mathbf{v}) = 0$ on $J_0^{1,4}$. It is well known, see for example [Gal94], that every continuous linear functional on $W_0^{1,4}$ vanishing on $J_0^{1,4}$ has a representation of the form

$$\int_\Omega \pi \; (\nabla \cdot \mathbf{v}) \, d\mathbf{x}$$

for some function $\pi \in L^{\frac{4}{3}}$ such that $\int_\Omega \pi \, d\mathbf{x} = 0$. So we are assured that there exists a function $\pi \in L^{\frac{4}{3}}$ such that

$$\int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u} \partial_j v_i + \pi \; (\nabla \cdot \mathbf{v}) \, d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle = 0 \tag{2.8}$$

for all $v \in W_0^{1,4}$.

At this point we should point out a significant difference between weak solutions of the Stokes system proper and the nonlinear Stokes-like system considered here. The weak solution of the Stokes system is found in the space $W_0^{1,2}(\Omega)$ which is a larger space than the one containing solutions of the nonlinear system, $J_0^{1,4}$. This phenomenon is what allows us, for example, to show unique weak solutions of the non-steady system (1.1). However, the pressure for the Stokes system lies in $L^2(\Omega)$ as opposed to the pressure appearing in the nonlinear system which lies in $L^{\frac{4}{3}}(\Omega)$. Thus the pressure in the nonlinear system apparently comes from a less regular space. All of this arises from balancing terms in the weak formulation. If $\mathbf{u}$ is an an arbitrary element of $J_0^{1,4}(\Omega)$, then $2(\nu_1 + \nu_2 |\mathbf{Du}|^2) D_{ij}\mathbf{u}$ is in $L^{\frac{4}{3}}$. So we cannot expect that π with which it is balanced lies in any smaller space. This balancing also allows us to have right hand sides $\mathbf{f}$ to be in $\left(J_0^{1,4}\right)^*$ which is in a bigger space than the one containing right hand sides for the Stokes system, $\left(J_0^{1,2}\right)^*$. Indeed, since we normally specify $\mathbf{f}$ to obtain $\mathbf{u}$, we can see that the less regular pressure is related to the fact that we are able to specify a less regular right hand side. The structure of the PDE determines spaces in which both solutions and legitimate right hand sides can be found.

The less regular space that contains the pressure becomes significant later on when we attempt to prove regularity for solutions of system (2.1). It turns out that a natural way to prove regularity for the Stokes system [SŠ73] relies heavily that the pressure does not live in any worse a space than L^2. Therefore, we will not be able to extend these ideas to the nonlinear Stokes-like system. Indeed, in the end we will obtain regularity only by avoiding the pressure completely. However, before we do this, we first study how classical techniques are applied to a model system similar to the nonlinear Stokes-like system.

Chapter 3
Existence of Second Derivatives for a Model System

In this Chapter we will examine the system:

$$
\begin{aligned}
-\partial_j \left(\left(\nu_1 + \nu_2 \left| \nabla \mathbf{u} \right|^2 \right) \partial_j u_i \right) &= f_i \\
\mathbf{u}|_{\partial\Omega} &= 0
\end{aligned} \tag{3.1}
$$

It is obviously similar to that studied in Chapter 2 , and we will use it as a model to investigate how to prove regularity for the nonlinear Stokes-like system (2.1). There are two differences between the model, which we shall call the nonlinear Poisson-like system, and the nonlinear Stokes-like system. Firstly, we have relaxed the constraint that solutions be solenoidal. This, in turn, increases the set of test functions to work with in the variational formulation. Secondly, we have replaced occurrences of the deformation tensor with the gradient tensor. This is necessary to preserve the nature of the nonlinearity; we want it to force solutions to be in $W_0^{1,4}$, just as in Chapter 2 .

By similar techniques as used in Chapter 2 we can show that if $\mathbf{f} \in \left(W_0^{1,4}\right)^*$ there exists a unique weak solution $\mathbf{u} \in W_0^{1,4}$ of (3.1), i.e. a function $\mathbf{u} \in W_0^{1,4}$ that satisfies

$$
\int_\Omega \left(\nu_1 + \nu_2 \left| \nabla \mathbf{u} \right|^2 \right) \partial_i u_j \partial_i \phi_j \, d\mathbf{x} = \langle \mathbf{f}, \phi \rangle \tag{3.2}
$$

for all $\phi \in C_0^\infty(\Omega)$. We have the following estimate for the size of such a solution:

$$
||\mathbf{u}||_{W_0^{1,4}} \leq \left(\frac{c(\Omega)}{\nu_2} ||\mathbf{f}||_{\left(W_0^{1,4}\right)^*} \right)^{\frac{1}{3}} . \tag{3.3}
$$

We now want to show that we have some greater regularity of the solution if we assume some greater regularity from $\mathbf{f}$. In particular, we will assume that $\mathbf{f} \in L^2$.

The techniques and results in this Chapter are not new. For example, the ideas we are about to use can be found in standard texts: for elliptic equations in [LU68] and for elliptic systems in [Neč83]. Actually, the structure of the system allows us to treat the system in the same fashion as an elliptic equation. We use the structure further to prove results directly with the idea of motivating the generalizations to come in Chapter 4 . Moreover, the intrinsic calculations are more complicated for the Stokes-like system and it will be useful to see the preliminary ideas in an isolated context. We will show in this Chapter how the theory of difference quotients can be used to attack nonlinear problems. By seeing how the standard theory works in this case, we will also be able to see how its direct application fails in the case of the nonlinear Stokes-like system and therefore why the results of Chapter 4 are interesting.

3.1 Difference Quotients

It seems reasonable at this point to give a very brief recollection of the fundamentals of difference quotients. The proofs of these results are well known and can be found, for example, in [Gia93]. Given a function g on $\mathbf{R}^3$, let $\tau_{h,m}(g)$ denote the quantity

$$\frac{g(\mathbf{x} + h\mathbf{e}_m) - g(\mathbf{x})}{h}.$$

If g is defined on our bounded set Ω, we extend for simplicity g by 0 to $\mathbf{R}^3$ to define the difference quotient in this case. Naturally the difference quotient in some sense approximates the derivative of a function. This intuition can be made rigorous as is seen by the following Lemma.

Lemma 3.1 *If $g \in W^{1,p}(\Omega)$ and Ω' is an open set with closure contained in Ω then there is a* *constant $c(\Omega, \Omega')$ such that*

$$||\tau_{h,m} g||_{L^p(\Omega')} \leq c(\Omega, \Omega')||\partial_m g||_{L^p(\Omega')}.$$

for all h such that $h < d(\partial\Omega', \partial\Omega)$. An upper bound for $c(\Omega, \Omega')$ is the number of cubes with length $\frac{d(\partial\Omega', \partial\Omega)}{2}$ required to cover $\bar{\Omega}'$, and therefore it depends only in this way on the regularity of Ω.

The converse of this Lemma is also true and is indeed more useful for our purposes.

Lemma 3.2 *If $g \in L^p(\Omega)$ and Ω' is an open set with closure contained in Ω and*

$$||\tau_{h,m} g||_p \leq k$$

for some fixed k and for all $h < d(\partial\Omega', \partial\Omega)$ then $g \in W^{1,p}(\Omega')$. Moreover,

$$||\partial_m g||_p \leq k$$

and $\tau_{h,m} g$ converges strongly in $L^p(\Omega')$ to $\partial_m g$.

Derivatives and difference quotients commute. Indeed, if $d(\partial\Omega', \partial\Omega) > 2h$ then

$$\tau_{h,m} g \in W^{1,p}(\Omega')$$

and $\partial_j \tau_{h,m} g = \tau_{h,m} \partial_j g$. The difference quotient also enjoys some properties analogous to those of derivatives. We have a rule for "integration by parts". If $g \in L^p(\Omega)$ and $f \in L^{p'}(\Omega)$ and either g or f have compact support then

$$\int_\Omega g \tau_{h,m} f \, d\mathbf{x} = -\int_\Omega \tau_{h,m} g f \, d\mathbf{x}$$

for h sufficiently small. We also have a "Leibniz rule",

$$\tau_{h,m}(fg(\mathbf{x})) = g(\mathbf{x})\tau_{h,m} f(\mathbf{x}) + f(\mathbf{x} + h\mathbf{e}_m)\tau_{h,m} g(\mathbf{x}).$$

3.2 Interior Regularity

Before we start the calculation to derive interior regularity, let us first motivate it with the *a priori* estimate that underlies it. Let $\mathbf{x}_0$ be an interior point of Ω and let η be a cut-off function in a neighbourhood of $\mathbf{x}_0$, so $\eta \in C_0^\infty(\Omega)$, $\eta(\mathbf{x}) \in [0,1]$ for all $\mathbf{x} \in \Omega$, and there exists a neighbourhood Ω' of $\mathbf{x}_0$ such that $\eta(\mathbf{x}) = 1$ in this neighbourhood. Assuming $\mathbf{u}$ is a smooth function, we can use $\partial_k(\eta^2 \partial_k \mathbf{u})$as a test function in (3.1). Doing this yields, after expanding some derivatives,

$$\int_\Omega f_i \partial_k(\eta^2 \partial_k u_i)\, d\mathbf{x} = -\int_\Omega \left(\nu(\nabla \mathbf{u})\partial_i\partial_k u_j + \mathbf{u}_2 \partial_r\partial_k u_s \partial_r u_s\right)\left(2\eta\partial_i\eta\partial_k u_j + \eta^2 \partial_i\partial_k u_j\right)\, d\mathbf{x}$$

and therefore

$$\begin{aligned}\int_\Omega \eta^2 \nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\partial_k u_j\, d\mathbf{x} = &-\int_\Omega \eta^2 \nu_2 \partial_r\partial_k u_s \partial_r u_s \partial_i\partial_k u_j \partial_i u_j\, d\mathbf{x} - \\ &-\int_\Omega 2\eta\nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\eta\partial_k u_j\, d\mathbf{x} - \\ &-\int_\Omega 2\nu_2\eta\partial_r\partial_k u_s \partial_r u_s \partial_i u_j \partial_i \eta \partial_k u_j\, d\mathbf{x} - \\ &-\int_\Omega f_i\partial_k(\eta^2\partial_k u_i)\, d\mathbf{x} \end{aligned} \tag{3.4}$$

where $\mathbf{u}(\nabla \mathbf{u}) = \mathbf{u}_1 + \nu_2 |\nabla \mathbf{u}|^2$. Applying Hölder's inequality in the above yields easily

$$\int_\Omega \eta^2 \nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\partial_k u_j\, d\mathbf{x} \le c(\nu_1)\left[||\eta \mathbf{f}||^2 + |||\nabla\eta|\sqrt{\nu(\nabla \mathbf{u})}\nabla \mathbf{u}||^2\right]. \tag{3.5}$$

Since the left hand of (3.5) side bounds the second derivatives of $\mathbf{u}$, i.e. $\nu_1||\eta^2\nabla^2 \mathbf{u}|| \le \int_\Omega \eta^2 \nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\partial_k u_j\, d\mathbf{x}$, we have the desired *a priori* estimate. The key part of this calculation involved the ellipticity of the system which appeared via

$$\begin{aligned}\int_\Omega \eta^2 \partial_k\left(\nu(\nabla \mathbf{u})\partial_i u_j\right)\partial_i\partial_k u_j\, d\mathbf{x} &= \int_\Omega \eta^2 \nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\partial_k u_j\, d\mathbf{x} + \\ &\quad + \int_\Omega \eta^2 \nu_2 \partial_r\partial_k u_s \partial_r u_s \partial_i\partial_k u_j \partial_i u_j\, d\mathbf{x} \\ &\ge \int_\Omega \eta^2 \nu(\nabla \mathbf{u})\partial_i\partial_k u_j \partial_i\partial_k u_j\, d\mathbf{x}. \end{aligned} \tag{3.6}$$

We now want to use the ideas of the *a priori* estimate in the difference quotient context. Let Ω'' be an open set with compact support in Ω such that $\operatorname{supp}(\eta)$ and $\operatorname{supp}(\eta(\mathbf{x}-h\mathbf{e}_m)$ are contained in Ω'' for every $h < h_0$ for some $h_0 > 0$. If h is sufficiently small then $\eta^2\tau_{h,m}(\mathbf{u}) \in W_0^{1,4}$ and so is $\tau_{-h,m}(\eta^2\tau_{h,m}(\mathbf{u}))$ and it is therefore a legitimate test function. Letting $\phi = \tau_{-h,m}\left(\eta^2\tau_{h,m}\left(\mathbf{u}\right)\right)$ in (3.2) we obtain

$$\begin{aligned}\int_\Omega f_i\tau_{-h,m}\left(\eta^2\tau_{h,m}\left(u_i\right)\right)\,d\mathbf{x} &= \int_\Omega \left(\nu_1+\nu_2\left|\nabla\mathbf{u}\right|^2\right)\partial_i u_j \partial_i \tau_{-h,m}\left(\eta^2\tau_{h,m}\left(u_j\right)\right)\,d\mathbf{x}\\ &= -\int_\Omega \tau_{h,m}\left(\left(\nu_1+\nu_2\left|\nabla\mathbf{u}\right|^2\right)\partial_i u_j\right)\partial_i\left(\eta^2\tau_{h,m}\left(u_j\right)\right)\,d\mathbf{x}\end{aligned} \tag{3.7}$$

The real trick now is to see that the elliptic properties observed for derivatives persist in some sense for difference quotients. We want a bound for the right hand side of (3.7) similar to (3.6). The trick can be found in [LU68] for a scalar quasilinear elliptic equation and it applies here also. We are able to write the quantity

$$\tau_{h,m}\left(\left(\nu_1+\nu_2\left|\nabla\mathbf{u}\right|^2\right)\partial_i u_j\right) = \tau_{h,m}\left(\nu\left(\nabla\mathbf{u}\right)\partial_i u_j\right)$$

in a convenient form. For almost every $\mathbf{x}$,

$$\begin{aligned}\tau_{h,m}\left(\nu\left(\nabla\mathbf{u}\right)\partial_i u_j\right) &= \frac{1}{h}\int_0^1 \frac{\partial}{\partial r}\Big[\left(\nu\left(r\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)+(1-r)\,\nabla\mathbf{u}(\mathbf{x})\right)\right)\cdot\\ &\qquad\qquad \cdot\left(\partial_i\left(ru_j(\mathbf{x}+h\mathbf{e}_m)\,(1-r)\,u_j(\mathbf{x})\right)\right)\Big]\,dr\\ &= \Bigg[\Big[\nu_1+\frac{\nu_2}{3}(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2+\partial_r u_s(\mathbf{x}+h\mathbf{e}_m)\partial_r u_s(\mathbf{x})+\\ &\qquad\qquad +|\nabla\mathbf{u}(\mathbf{x})|^2)\Big]\delta_{ik}\delta_{jl}+\\ &\quad+\frac{2\nu_1}{3}\Big[\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right)+\frac{1}{2}\partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right)+\\ &\qquad +\frac{1}{2}\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\partial_i u_j\left(\mathbf{x}\right)+\partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}\right)\Big]\Bigg]\tau_{h,m}\left(\partial_k u_l\right)\\ &= a^{h,m}_{ij,kl}\tau_{h,m}\left(\partial_k u_l\right).\end{aligned} \tag{3.8}$$

With this explicit expression, we are able to write some estimates (boundedness and coercivity) for the bilinear form $a_{ik,jl}(h,m)$. For the coercivity estimate, we have

$$\begin{aligned}
a^{h,m}_{ij,kl}\xi_{kl}\xi_{ij} &= \Big(\nu_1 + \frac{\nu_2}{3}\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + \partial_r u_s(\mathbf{x}+h\mathbf{e}_m)\partial_r u_s(\mathbf{x}) + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\Big)|\boldsymbol{\xi}|^2 + \\
&\quad + \frac{2\nu_2}{3}\big(\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right) + \partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right) + \\
&\qquad\qquad + \partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}\right)\big)\xi_{kl}\xi_{ij} \\
&\geq \Big(\nu_1 + \frac{\nu_2}{6}\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\Big)|\boldsymbol{\xi}|^2 + \\
&\qquad + \frac{\nu_2}{3}\left(\left(\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\xi_{kl}\right)^2 + \left(\partial_k u_l\left(\mathbf{x}\right)\xi_{kl}\right)^2\right) \\
&\geq \frac{1}{6}\left(\nu_1 + \nu_2\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\right)|\boldsymbol{\xi}|^2 .
\end{aligned}$$

For the boundedness estimate, we have

$$\begin{aligned}
a^{hm}_{ik,jl}\xi_{kl}\xi_{ij} &= \Big(\nu_1 + \frac{\nu_2}{3}\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + \partial_r u_s(\mathbf{x}+h\mathbf{e}_m)\partial_r u_s(\mathbf{x}) + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\Big)|\boldsymbol{\xi}|^2 + \\
&\quad + \frac{2\nu_2}{3}\big(\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right) + \partial_l u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right) \\
&\qquad\qquad + \partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}\right)\big)\xi_{kl}\xi_{ij} \\
&\leq \Big(\nu_1 + \frac{\nu_2}{2}\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\Big)|\boldsymbol{\xi}|^2 \\
&\quad + \nu_2\left(\partial_k u_l\left(\mathbf{x}+h\mathbf{e}_m\right)\partial_i u_j\left(\mathbf{x}+h\mathbf{e}_m\right) + \partial_k u_l\left(\mathbf{x}\right)\partial_i u_j\left(\mathbf{x}\right)\right)\xi_{kl}\xi_{ij} \\
&\leq \Big(\nu_1 + \frac{\nu_2}{2}\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right) \\
&\qquad + \nu_2\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\Big)|\boldsymbol{\xi}|^2 \\
&\leq \frac{3}{2}\left(\nu_1 + \nu_2\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right)\right)|\boldsymbol{\xi}|^2 .
\end{aligned}$$

Since we will encounter this expression several times, let us define

$$\mu_{hm} = \nu_1 + \nu_2\left(|\nabla\mathbf{u}(\mathbf{x}+h\mathbf{e}_m)|^2 + |\nabla\mathbf{u}(\mathbf{x})|^2\right).$$

In summary, then, we have shown, that

$$\frac{1}{6}\mu_{h,m}|\boldsymbol{\xi}|^2 \quad\leq\quad a^{h,m}_{ij,kl}\xi_{kl}\xi_{ij} \quad\leq\quad \frac{3}{2}\mu_{h,m}|\boldsymbol{\xi}|^2 . \tag{3.9}$$

Now we use these estimates to analyse (3.7). From (3.7) and (3.8) we obtain

$$\begin{aligned}\int_\Omega f_i \tau_{-h,m}(\eta^2 \tau_{h,m}(u_i))\, d\mathbf{x} &= -\int_\Omega a_{ij,kl}^{h,m} \partial_k \tau_{h,m}(u_l)\, \partial_i \left(\eta^2 \tau_{h,m}(u_j)\right)\, d\mathbf{x} \\ &= -\int_\Omega \eta^2 a_{ij,kl}^{h,m} \partial_k \tau_{h,m}(u_l)\, \partial_i \left(\tau_{h,m}(u_j)\right)\, d\mathbf{x} - \\ &\quad - \int_\Omega a_{ij,kl}^{h,m} \partial_k \tau_{h,m}(u_l)\, 2\eta \partial_i \eta \left(\tau_{h,m}(u_j)\right)\, d\mathbf{x},\end{aligned}$$

so

$$\begin{aligned}\int_\Omega \eta^2 a_{ij,kl}^{h,m} \partial_k \tau_{h,m}(u_l)\, \partial_i \left(\tau_{h,m}(u_j)\right)\, d\mathbf{x} &= -\int_\Omega f_j \tau_{-h,m}(\eta^2 \tau_{h,m}(u_j))\, d\mathbf{x} - \\ &\quad - \int_\Omega a_{ij,kl}^{h,m} \partial_k \tau_{h,m}(u_l)\, 2\eta \partial_i \eta \left(\tau_{h,m}(u_j)\right)\, d\mathbf{x}.\end{aligned} \tag{3.10}$$

We now want to estimate from above both terms on the right hand side. Let us start with the first of the two:

$$\begin{aligned}\left|\int_\Omega f_j \tau_{-h,m}(\eta^2 \tau_{h,m}(u_j))\, d\mathbf{x}\right| &= \left|\int_\Omega f_j \left[\tau_{-h,m}(\eta)\eta\tau_{h,m}(u_j) + \eta\left(\mathbf{x} - h\mathbf{e}_m\right)\tau_{-h,m}(\eta\tau_{h,m}(u_j))\right] d\mathbf{x}\right| \\ &\leq ||\eta f||_{L^2} ||\tau_{-h,m}(\eta)\tau_{h,m}(\mathbf{u})||_{L^2} + \\ &\quad + ||f\eta(\mathbf{x} - h\mathbf{e}_m)||_{L^2} ||\tau_{-h,m}(\eta\tau_{h,m}(\mathbf{u}))||_{L^2} \\ &\leq ||\eta f||_{L^2} ||\tau_{-h,m}(\eta)\tau_{h,m}(\mathbf{u})||_{L^2} + \frac{1}{\epsilon' \nu_1} ||\eta(\mathbf{x} + h\mathbf{e}_m) f||_{L^2}^2 + \\ &\quad + \nu_1 \epsilon' ||\tau_{-h,m}(\eta\tau_{h,m}(\mathbf{u}))||_{L^2}^2.\end{aligned} \tag{3.11}$$

At this point we invoke Lemma 3.1 to bound the last term here. From Lemma 3.1 we see that there exists $c(\Omega, \Omega'')$ such that for all $\mathbf{v}$ in $W_0^{1,4}(\Omega'')$,

$$||\tau_{h,m}(\mathbf{v})||_{L^p)} \leq c(\Omega, \Omega'')||\nabla \mathbf{v}||_{L^p}.$$

From this we obtain

$$
\begin{aligned}
||\tau_{-h,m}(\eta\tau_{h,m}(\mathbf{u}))||_{L^2}^2 &\leq c(\Omega,\Omega'')||\nabla(\eta\tau_{h,m}(\mathbf{u}))||_{L^2}^2 \\
&\leq c(\Omega,\Omega'')\left(||\nabla(\eta)\tau_{h,m}(\mathbf{u})||_{L^2}^2 + ||\eta\nabla\tau_{h,m}(\mathbf{u})||_{L^2}\right)^2 \\
&\leq c(\Omega,\Omega'')\left(||\nabla(\eta)\tau_{h,m}(\mathbf{u})||_{L^2}^2 + \frac{1}{\nu_1}|\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(\mathbf{u})||_{L^2}^2\right) \\
&\leq c(\Omega,\Omega'')\left(||\nabla\eta\tau_{h,m}\mathbf{u}||_{L^2}^2 + \frac{1}{\nu_1}||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(\mathbf{u})||_{L^2}^2\right) \qquad (3.12)
\end{aligned}
$$

Combining (3.11) and (3.12), letting $\epsilon = \epsilon' c(\Omega,\Omega'')$ we obtain the final estimate for this term,

$$
\begin{aligned}
\left|\int_\Omega f_j\tau_{-h,m}(\eta^2\tau_{h,m}(u_j))\,d\mathbf{x}\right| \leq c(\Omega,\Omega'')\Big[\left(1+\frac{1}{\epsilon\nu_1}\right)||(\eta+\eta(\mathbf{x}+h\mathbf{e}_m))f||_{L^2}^2 + \\
+(1+\nu_1\epsilon)|(\nabla\eta+\tau_{h,m}\eta)\tau_{h,m}\mathbf{u}||_{L^2}^2\Big] + \epsilon||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(u)||_{L^2}.
\end{aligned}
\qquad (3.13)
$$

Now we turn to the second term on the right hand side of (3.10). It is not hard to show using the form of $\mathbf{a}^{h,m}$ and the Cauchy-Schwarz inequality that

$$
\left|a_{ij,kl}^{h,m}\xi_{ij}\chi_{kl}\right| \leq \left(a_{ij,kl}^{h,m}\xi_{ij}\xi_{kl}\right)^{\frac{1}{2}}\left(a_{ij,kl}^{h,m}\chi_{ij}\chi_{kl}\right)^{\frac{1}{2}}. \qquad (3.14)
$$

Indeed, $\mathbf{a}^{h,m}$ is a positive definite symmetric bilinear form on the vector space of n by n matrices, so (3.14) is just an assertion of Cauchy-Schwarz. Using (3.14) in the second term on the right hand side of (3.10) we obtain

$$\begin{aligned}\left|\int_\Omega a^{h,m}_{ij,kl}\partial_k\tau_{h,m}\left(u_l\right)\eta\partial_i\eta\left(\tau_{h,m}\left(u_j\right)\right)\,d\mathbf{x}\right| &\leq \int_\Omega\left(\eta^2 a^{h,m}_{ij,kl}\partial_k\tau_{h,m}u_l\partial_i\tau_{h,m}u_j\right)^{\frac{1}{2}}\cdot\\ &\quad\cdot\left(a^{h,m}_{ij,kl}\tau_{h,m}\left(u_j\right)\tau_{h,m}\left(u_l\right)\partial_j\eta\partial_k\eta\right)^{\frac{1}{2}}\,d\mathbf{x}\\ &\leq \frac{2\epsilon}{3}\int_\Omega\eta^2 a^{h,m}_{ij,kl}\partial_k\tau_{h,m}u_l\partial_i\tau_{h,m}u_j\,d\mathbf{x}+\\ &\quad+\frac{c}{\epsilon}\int_\Omega a^{h,m}_{ij,kl}\tau_{h,m}\left(u_j\right)\tau_{h,m}\left(u_l\right)\partial_j\eta\partial_k\eta\,d\mathbf{x}\\ &\leq \epsilon||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(u)||^2_{L^2}+\\ &\quad+\frac{c}{\epsilon}\int_\Omega\mu_{h,m}|\nabla\eta|^2|\tau_{h,m}u|^2\,d\mathbf{x}.\end{aligned} \tag{3.15}$$

On the other hand, we can bound below the left hand side of (3.10) using (3.9) to get

$$\frac{1}{6}||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(u)||^2_{L^2}\leq\int_\Omega\eta^2 a^{h,m}_{ij,kl}\partial_k\left(\tau_{h,m}\left(u_l\right)\right)\partial_i\left(\tau_{h,m}\left(u_j\right)\right)\,d\mathbf{x}. \tag{3.16}$$

Letting $\epsilon = 1/24$ we can combine the lower estimate (3.16) with the upper estimates (3.13) and (3.15) for equation (3.10) to get

$$\begin{aligned}||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}\mathbf{u}||^2_{L^2}\leq c(\Omega',\Omega,\nu_1)\Big[\int_\Omega|\nabla\eta|^2\mu_{h,m}|\tau_{h,m}u|^2\,d\mathbf{x}+||(\eta+\eta(\mathbf{x}+h\mathbf{e}_m))f||^2_{L^2}+\\ +||(\nabla\eta+\tau_{h,m}\eta)\tau_{h,m}\mathbf{u}||^2_{L^2}\Big].\end{aligned} \tag{3.17}$$

The right hand side of (3.17) is uniformly bounded in h. Indeed, by the absolute continuity of the integral we see $\mu_{h,m}$ converges strongly in L^2 to $\mu_{0,m} = \nu_1 + 2\nu_2|\nabla\mathbf{u}|^2$, which we will call μ. From Lemma 3.2 we know $\tau_{h,m}\mathbf{u}$ converges strongly in L^4 to $\partial_m\mathbf{u}$. Finally, from the smoothness of η we know $\tau_{h,m}\mathbf{u}$ converges uniformly to $\partial_k\eta$. From the boundedness of convergent sequences follows the boundedness of the right hand side of (3.17). Since

$$||\nabla\tau_{h,m}(u)||_{L^2(\Omega')}||\leq\frac{1}{\nu_1}||\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(u)||_{L^2}, \tag{3.18}$$

we see from the uniform bound in h of the right hand side of (3.17) that the left hand side of (3.18) is uniformly bounded in h. From Lemma 3.2, then, we assert that $\partial_m\partial_i u$

exists in $L^2(\Omega')$ for each i. Since m is arbitrary, $u \in W^{2,2}(\Omega')$, and since x_0 is arbitrary, we have $u \in W^{2,2}(\Omega')$ for any $\Omega' \subset\subset \Omega$.

We would now like to get an estimate for the interior derivatives similar to that of the *a priori* estimate (3.5). To do this we note that from (3.17) and the uniform boundedness of the right hand side of (3.17), the sequence $\sqrt{\mu_{h,m}}\eta\nabla\tau_{h,m}(\mathbf{u})$ converges weakly to some limit. From the strong convergence of $\mu_{h,m}$ to μ in L^2 and the strong convergence of $\eta\nabla\tau_{h,m}(\mathbf{u})$ to $\eta\nabla\partial(\mathbf{u})$ in L^2 we see that the weak limit must be $\sqrt{\mu}\eta\nabla\partial_m\mathbf{u}$. Indeed, if ϕ is smooth we have from the aforementioned strong convergence

$$\lim_{h\to 0}\int_\Omega \sqrt{\mu_{h,m}}\eta\partial_i\tau_{h,m}u_j\phi_j\, d\mathbf{x} = \int_\Omega \sqrt{\mu}\eta\partial_i\partial_m u_j\phi_j\, d\mathbf{x}.$$

The result follows from the density of smooth functions in L^2. Since the norm of the weak limit is less than the limit infimum of the norms of the limiting terms we conclude that

$$||\sqrt{\mu}\eta\nabla\tau_{h,m}\mathbf{u}||_{L^2}^2 \leq \liminf_{h\to 0} c(\Omega',\Omega,\nu_1)\Big[\int_\Omega |\nabla\eta|^2\mu_{h,m}|\tau_{h,m}u|^2\, d\mathbf{x}+ + ||(\eta+\eta(\mathbf{x}+h\mathbf{e}_m))f||_{L^2}^2 + +||(\nabla\eta+\tau_{h,m}\eta)\tau_{h,m}\mathbf{u}||_{L^2}^2\Big]. \tag{3.19}$$

From the strong convergence of $\tau_{h,m}\mathbf{u}$ to $\partial_m\mathbf{u}$ in L^4 and the strong convergence of $\mu_{h,m}$ to μ in L^2 as well as the uniform convergence of $\eta(\mathbf{x}+h\mathbf{e}_m)$ and $\tau_{h,m}\eta$ to η and $\partial_m\eta$ respectively we can take the limit in the right hand side of (3.19) to conclude

$$||\sqrt{\mu}\eta\nabla\partial_m\mathbf{u}||_{L^2}^2 \leq c(\Omega',\Omega,\nu_1)\Big[\int_\Omega |\nabla\eta|^2\mu|\partial_m\mathbf{u}|^2\, d\mathbf{x} + ||\eta\mathbf{f}||_{L^2}^2\Big]. \tag{3.20}$$

Indeed this is almost what we obtained in the *a priori* estimate (3.5). We would want to remove the dependence of the constant on Ω and express the interior estimate only in terms of the cut-off function η. This can be done as we will see in Chapter 4, but we need not do this for now for our purposes.

3.3 Boundary Regularity

Although we will not be able to prove boundary regularity for the Stokes-like problem (2.1), we will include boundary regularity for the Poisson-like system to complete the ideas of this Chapter. We will assume for this section that $\partial\Omega$ is of class $C^{1,1}$. Let $\mathbf{x}_0$ be a point on $\partial\Omega$, so there exists a $C^{1,1}$ invertible function $\mathbf{G}$ mapping B_R^+ to Ω', where Ω' is the intersection of some neighbourhood of $\mathbf{x}_0$ and Ω. If v is a function on Ω', let v^* denote the pull back of v, so $v^*(\mathbf{y}) = v(\mathbf{G}(\mathbf{y}))$. Since $\mathbf{G}$ is $C^{1,1}$, if $v \in W^{1,4}(\Omega')$ then $v^* \in W^{1,4}(B_R^+)$. Let $\mathbf{J}$ be the Jacobian of $\mathbf{G}$, so $\nabla_y \mathbf{u}^* = \nabla_x \mathbf{u}\mathbf{J}$. We define $\mathbf{U}$ to be $(\nabla_y \mathbf{u}^*)\,\mathbf{J}^{-1}$. Already this section will be notationally clumsy, so we will introduce $\mathbf{A} : \mathbf{B}$ to mean $A_{ij}B_{ij}$ hoping to make some of the calculations more clear. Let us suppose that $\Phi \in W_0^{1,4}(B_R^+)$, and let $\phi(\mathbf{x}) = \Phi(\mathbf{G}^{-1}(\mathbf{x}))$. Using this ϕ in (3.2) and changing variables we obtain

$$\begin{aligned}
\int_{B_R^+} |\det \mathbf{J}| \mathbf{f}_i^* \Phi_i \, d\mathbf{y} &= \int_\Omega f_i \phi_i \, d\mathbf{y} \\
&= \int_\Omega \left(\nu_1 + \nu_2 \left|\nabla_x \mathbf{u}\right|^2\right) \nabla_x \mathbf{u} : \nabla_x \phi \, d\mathbf{y} \\
&= \int_{B_R^+} |\det \mathbf{J}| \left(\nu_1 + \nu_2 \left|\mathbf{U}\right|^2\right) \mathbf{U} : \nabla_y \Phi \mathbf{J}^{-1} \, d\mathbf{y}. \qquad (3.21)
\end{aligned}$$

We can now use techniques similar to those in the previous section. As before we take η to be a cut-off function with support in B_R such that $\eta|_{\mathbf{y}+h\mathbf{e}_m}$ also has support in B_R.

Letting $\Phi = \tau_{-h,m}(\eta^2 \tau_{h,m}(u^*))$ in the above where $1 \leq m < n$ we obtain

$$\begin{aligned}
&\int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{U}\right)_{kl} \eta^2 \, d\mathbf{y} = \\
&\qquad \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\nabla \mathbf{u}^*(\mathbf{y}+h\mathbf{e}_m)\tau_{h,m}\mathbf{J}^{-1}\right)_{kl} \eta^2 \, d\mathbf{y} - \\
&\qquad - \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{u}^*\mathbf{J}^{-1}\right)_{kl} 2\eta \nabla \eta \, d\mathbf{y} + \\
&\qquad + \int_{B_R^+} F_{ij}^{h,m} \left[\eta^2 \left(\nabla \mathbf{u}^* \big|_{\mathbf{y}+h\mathbf{e}_m} \tau_{h,m}\mathbf{J}^{-1} \right)_{ij} - \eta^2 \tau_{h,m} \left(\nabla \mathbf{u}^* \mathbf{J}^{-1}\right)_{ij} \right] d\mathbf{y} - \\
&\qquad - \int_{B_R^+} |\det \mathbf{J}| f_i \tau_{-h,m} \left(\eta^2 \tau_{h,m} u_i^*\right) \, d\mathbf{y} - \\
&\qquad - \int_{B_R^+} |\det \mathbf{J}| \nu(\mathbf{U}) \mathbf{U} : \left(\tau_{h,m} \ \mathbf{u}^* \big|_{\mathbf{y}-h\mathbf{e}_m} 2\nabla \eta \tau_{-h,m}\mathbf{J}^{-1} \right) \eta \, d\mathbf{y} - \\
&\qquad - \int_{B_R^+} |\det \mathbf{J}| \nu(\mathbf{U}) \mathbf{U} : \left(\tau_{h,m} \ \nabla \mathbf{u}^* \big|_{\mathbf{y}-h\mathbf{e}_m} \tau_{-h,m}\mathbf{J}^{-1} \right) \eta^2 \, d\mathbf{y}. \qquad (3.22)
\end{aligned}$$

Here we have used the notation

$$\begin{aligned}
a_{ij,kl}^{h,m} = \int_0^1 |\det \mathbf{J}(\mathbf{y} + rh\tau_{h,m}\mathbf{y})| \big[&\nu(\mathbf{U} + rh\tau_{h,m}\mathbf{U})\delta_{ik}\delta_{jl} + \\
&\nu_2(\mathbf{U} + rh\tau_{h,m}\mathbf{U})_{ij}(\mathbf{U} + rh\tau_{h,m}\mathbf{U})_{kl} \big] \, dr \qquad (3.23)
\end{aligned}$$

and

$$F_{ij}^{h,m} = \int_0^1 \partial_m \, |\det \mathbf{J}|_{\mathbf{y}+rh\tau_{h,m}\mathbf{y}} \, \nu(\mathbf{U} + rh\tau_{h,m}\mathbf{U})(\mathbf{U} + rh\tau_{h,m}\mathbf{U})_{ij} \, dr.$$

Since $|\det \mathbf{J}|$ is bounded both from above and away from zero, we have the same estimates as the previous section for $\mathbf{a}^{h,m}$, namely

$$c(\partial\Omega)\mu_{h,m} |\boldsymbol{\xi}|^2 \quad \leq \quad a_{ij,kl}^{h,m}\xi_{ij}\xi_{kl} \quad \leq \quad c(\partial\Omega)\mu_{h,m} |\boldsymbol{\xi}|^2 \qquad (3.24)$$

where this time

$$\mu_{h,m} = \nu_1 + \nu_2 \left(|\mathbf{U}(\mathbf{y} + h\mathbf{e}_m)|^2 + |\mathbf{U}|^2 \right).$$

As well as $\mathbf{a}^{h,m}$ we can also estimate $\mathbf{F}^{h,m}$. For any $g_{ij} \in W_0^{1,4}(B_R^+)$ we have, using the fact that $|\det \mathbf{J}|$ is bounded away from zero and that $\mathbf{J}$ is Lipschitz,

$$\int_{B_R^+} F_{ij}^{h,m} g_{ij}\, d\mathbf{y} \leq \int_{B_R^+} \int_0^1 |\partial_m |\det \mathbf{J}(\mathbf{y}+rh\tau_{h,m}\mathbf{y})|| \, \nu(\mathbf{U}+rh\tau_{h,m}\mathbf{U}) \, |\mathbf{U}+rh\tau_{h,m}\mathbf{U}| \, |\mathbf{g}| \, dr d\mathbf{y}$$
$$\leq \frac{c(\partial\Omega)}{\epsilon} \int_{B_R^+} \mu_{h,m} \left(\left| \mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m} \right| + |\mathbf{U}|^2 \right) d\mathbf{y} + \epsilon \int_{B_R^+} \mu_{h,m}^2 |\mathbf{g}|^2 \, d\mathbf{y}. \tag{3.25}$$

In the right hand side of (3.22) we now have six terms to estimate. Let us make the estimates for each term in order denoting each integral by a Roman numeral from I to VI.

For the first we have

$$\begin{aligned}
|\mathrm{I}| &= \left| \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \, \tau_{h,m}\mathbf{J}^{-1} \right)_{kl} \eta^2 \, d\mathbf{y} \right| \\
&\leq \epsilon \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{U}\right)_{kl} \eta^2 \, d\mathbf{y} + \\
&\quad + \frac{c}{\epsilon} \int_{B_R^+} a_{ij,kl}^{h,m} \left(\nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \, \tau_{h,m}\mathbf{J}^{-1} \right)_{ij} \left(\nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \, \tau_{h,m}\mathbf{J}^{-1} \right)_{kl} \eta^2 \, d\mathbf{y} \\
&\leq \epsilon \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{U}\right)_{kl} \eta^2 \, d\mathbf{y} + \frac{c(\partial\Omega)}{\epsilon} \int_{B_R^+} \mu_{h,m} \left| \nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \right|^2 \eta^2 \, d\mathbf{y}. \qquad (3.26)
\end{aligned}$$

The second term is estimated similarly. For it we have

$$\begin{aligned}
|\mathrm{II}| &= \left| 2 \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{u}^*\mathbf{J}^{-1}\right)_{kl} \eta \nabla \eta \, d\mathbf{y} \right| \\
&\leq \epsilon \int_{B_R^+} a_{ij,kl}^{h,m} \left(\tau_{h,m}\mathbf{U}\right)_{ij} \left(\tau_{h,m}\mathbf{U}\right)_{kl} \eta^2 \, d\mathbf{y} + \frac{c(\partial\Omega)}{\epsilon} \int_{B_R^+} \mu_{h,m} |\tau_{h,m}\mathbf{u}^*|^2 \, d\mathbf{y}. \qquad (3.27)
\end{aligned}$$

From our estimate (3.25) for $\mathbf{F}^{h,m}$ we can bound the third term by

$$\begin{aligned}
|\mathrm{III}| &= \left| \int_{B_R^+} F_{ij}^{h,m} \left[\eta^2 \left(\nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \, \tau_{h,m}\mathbf{J}^{-1} \right)_{ij} - \eta^2 \tau_{h,m} \left(\nabla \mathbf{u}^* \mathbf{J}^{-1} \right)_{ij} \right] d\mathbf{y} \right| \\
&\leq c(\partial\Omega) \left(1 + \frac{1}{\epsilon} \right) \int_{B_R^+} \mu_{h,m} \left(\left| \mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m} \right| + |\mathbf{U}|^2 \right) \eta^2 \, d\mathbf{y} + \\
&\quad + c(\partial\Omega) \int_{B_R^+} \mu_{h,m} \eta^2 \left| \nabla \mathbf{u}^*|_{\mathbf{y}+h\mathbf{e}_m} \right|^2 d\mathbf{y} + \epsilon \int_{B_R^+} \mu_{h,m} \eta^2 \left| \tau_{h,m} \left(\nabla \mathbf{u}^* \mathbf{J}^{-1} \right) \right|^2 d\mathbf{y}.
\end{aligned}$$
$$(3.28)$$

The fourth term is estimated as before in (3.13). We arrive at

$$\begin{aligned}|\mathrm{IV}| &= \left|\int_{B_R^+} |\det \mathbf{J}| f_i \tau_{-h,m}\left(\eta^2 \tau_{h,m} u_i^*\right)\, d\mathbf{y}\right| \\ &\le c(\partial\Omega)\left[\left(1+\frac{1}{\epsilon\nu_1}\right)\int_{B_R^+} |\det \mathbf{J}||\mathbf{f}|^2\, d\mathbf{y} + (1+\nu_1\epsilon)\, d\mathbf{y}\int_{B_R^+} |\det \mathbf{J}||\nabla \mathbf{u}^*|^2\, d\mathbf{y}\right]+ \\ &\quad + \epsilon\int_{B_R^+} \mu_{h,m}\eta^2\left|\tau_{h,m}\nabla \mathbf{u}^*\right|^2\, d\mathbf{y}. \end{aligned} \tag{3.29}$$

Since $\tau_{h,m}\nabla\mathbf{u}^* = \tau_{h,m}(\mathbf{U})\mathbf{J} - \left.\nabla\mathbf{u}^*\right|_{\mathbf{y}+h\mathbf{e}_m}\tau_{h,m}(\mathbf{J}^{-1})\mathbf{J}$ we can estimate the last term with

$$\epsilon\int_{B_R^+}\mu_{h,m}\eta^2\left|\tau_{h,m}\nabla\mathbf{u}^*\right|^2 \le 2\epsilon\int_{B_R^+}\mu_{h,m}\eta^2\left|\tau_{h,m}\mathbf{U}\right|^2\, d\mathbf{y} + c(\partial\Omega)\epsilon\int_{B_R^+}\mu_{h,m}\eta^2\left|\nabla\mathbf{u}^*\right|^2_{\mathbf{y}+h\mathbf{e}_m}\, d\mathbf{y}. \tag{3.30}$$

Combining (3.29) and (3.30) yields our final estimate for the fourth term:

$$\begin{aligned}|\mathrm{IV}| &\le c(\partial\Omega)\left[\left(1+\frac{1}{\epsilon\nu_1}\right)\int_{B_R^+}|\det J| f^2\, d\mathbf{y} + (1+\nu_1\epsilon)\int_{B_R^+}|\nabla u^*|^2\, d\mathbf{y}\right]+ \\ &\quad + c(\partial\Omega)\epsilon\int_{B_R^+}\mu_{h,m}\eta^2\left|\left.\nabla\mathbf{u}^*\right|_{\mathbf{y}+h\mathbf{e}_m}\right|^2\, d\mathbf{y} + 2\epsilon\int_{B_R^+}\mu_{h,m}\eta^2\left|\tau_{h,m}\mathbf{U}\right|^2\, d\mathbf{y}. \end{aligned} \tag{3.31}$$

Using Hölder's inequality and the fact that $|\det\mathbf{J}|$ is bounded away from zero and that the derivatives of $\mathbf{J}^{-1}$ are in L^∞ we easily estimate the fifth term by

$$\begin{aligned}|\mathrm{V}| &= \left|\int_{B_R^+}|\det\mathbf{J}|\nu(\mathbf{U})\mathbf{U} : \left(\tau_{h,m}\left.\mathbf{u}^*\right|_{\mathbf{y}-h\mathbf{e}_m}\nabla\eta\tau_{h,m}\mathbf{J}^{-1}\right)\eta\, d\mathbf{y}\right| \\ &\le c(\partial\Omega)\left[\int_{B_R^+}\nu(\mathbf{U})\mathbf{U} : \mathbf{U}\eta^2\, d\mathbf{y} + \int_{B_R^+}\nu(\mathbf{U})\left|\tau_{h,m}\mathbf{u}^*\right|^2_{\mathbf{y}-h\mathbf{e}_m}\, d\mathbf{y}\right]. \end{aligned} \tag{3.32}$$

The last term is mildly more tricky. Since $\tau_{h,m}\nabla\mathbf{u}^* = \tau_{h,m}(\mathbf{U})\mathbf{J} - \left.\nabla\mathbf{u}^*\right|_{\mathbf{y}+h\mathbf{e}_m}$ we have

$$\begin{aligned}|\mathrm{VI}| &= \left|\int_{B_R^+}|\det\mathbf{J}|\nu(\mathbf{U})\mathbf{U} : \left(\tau_{h,m}\left.\nabla\mathbf{u}^*\right|_{\mathbf{y}-h\mathbf{e}_m}\tau_{h,m}\mathbf{J}^{-1}\right)\eta^2\, d\mathbf{y}\right| \\ &\le \frac{c}{\epsilon}\int_{B_R^+}|\det\mathbf{J}|\nu(\mathbf{U})\mathbf{U} : \mathbf{U}\eta^2\, d\mathbf{y} + c(\partial\Omega)\epsilon\int_{B_R^+}\nu(\mathbf{U})|\nabla\mathbf{u}^*|^2\eta^2\, d\mathbf{y} + \\ &\quad + \epsilon\int_{B_R^+}\nu(\mathbf{U})\left|\tau_{h,m}(\left.\mathbf{U}\right|_{\mathbf{y}-h\mathbf{e}_m})\right|^2\eta^2\, d\mathbf{y}. \end{aligned} \tag{3.33}$$

We want to be able to absorb this last term into the left hand side. Changing variables we have

$$\begin{aligned}\int_{B_R^+} \nu(\mathbf{U}) \left|\tau_{h,m}(\mathbf{U}|_{\mathbf{y}-h\mathbf{e}_m})\right|^2 \eta^2 \, d\mathbf{y} &= \int_{B_R^+} \nu(\mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m}) \left|\tau_{h,m}(\mathbf{U})\right|^2 \eta^2\big|_{\mathbf{y}+h\mathbf{e}_m} \, d\mathbf{y} \\ &\leq \int_{B_R^+} \nu(\mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m}) \left|\tau_{h,m}(\mathbf{U})\right|^2 \left(\eta^2 + (h \sup \nabla\eta)^2\right) \, d\mathbf{y} \\ &\leq \int_{B_R^+} \mu_{h,m} \left|\tau_{h,m}(\mathbf{U})\right|^2 \eta^2 \, d\mathbf{y} + \\ &\qquad + c(\eta) \int_{B_R^+} \mu_{h,m} \Big(\left|\mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m}\right|^2 + |\mathbf{U}|^2 \Big) \, d\mathbf{y}.\end{aligned} \tag{3.34}$$

Combining (3.33) with (3.34) we arrive at our last estimate for the sixth term,

$$\begin{aligned}|\mathrm{VI}| \leq \frac{c}{\epsilon} \int_{B_R^+} |\det \mathbf{J}| \nu(\mathbf{U}) \mathbf{U} : \mathbf{U} \eta^2 \, d\mathbf{y} + c(\partial\Omega)\epsilon \int_{B_R^+} \nu(\mathbf{U}) |\nabla \mathbf{u}^*|^2 \eta^2 \, d\mathbf{y} + \\ + c(\eta)\epsilon \int_{B_R^+} \mu_{h,m} \left(\left|\mathbf{U}|_{\mathbf{y}+h\mathbf{e}_m}\right|^2 \, d\mathbf{y} + |\mathbf{U}|^2 \right) \, d\mathbf{y} + \epsilon \int_{B_R^+} \mu_{h,m} \left|\tau_{h,m}(\mathbf{U})\right|^2 \eta^2 \, d\mathbf{y}.\end{aligned} \tag{3.35}$$

We now combine the lower bound for the left hand side given in (3.24) with the upper estimates in (3.26), (3.27), (3.28), (3.31), (3.32) and (3.35), absorbing terms on the right similar to those on the left. The remainder of the proof continues as in the interior case to get that $\mathbf{U}$ has square integrable derivatives in each of the directions $1 \leq m < n$ and

$$\begin{aligned}&\int_{B_R^+} \nu(\mathbf{U}) \left|\partial_m \mathbf{U}\right|^2 \, d\mathbf{y} \leq \\ &\qquad c(\partial\Omega, \nu_1) \left[\int_{B_R^+} \nu(\mathbf{U}) \left|\nabla \mathbf{u}^*\right|^2 \, d\mathbf{y} + \int_{B_R^+} \nu(\mathbf{U}) \left|\mathbf{U}\right|^2 \, d\mathbf{y} + \int_{B_R^+} |\det \mathbf{J}| |\mathbf{f}|^2 \, d\mathbf{y} \right].\end{aligned} \tag{3.36}$$

We now use the fact that $\mathbf{J}$ has determinant bounded above and below to get $|\mathbf{U}|$ has bounds by

$$c(\partial\Omega)|\nabla u^*| \leq |\mathbf{U}| \leq c(\partial\Omega)|\nabla u^*|$$

to get

$$\int_{B_R^+} \nu(\nabla \mathbf{u}^*) \left|\partial_m \nabla \mathbf{u}^*\right|^2 \, d\mathbf{y} \le c(\partial\Omega, \nu_1) \left[\int_{B_R^+} \nu(\nabla u^*) \left|\nabla u^*\right|^2 \, d\mathbf{y} + \int_{B_R^+} |\det \mathbf{J}| |\mathbf{f}|^2 \, d\mathbf{y} \right]. \tag{3.37}$$

We have thus shown the existence of and given estimates for $\partial_m \partial_k \mathbf{u}^*$ for $1 \le m < n$, $1 \le k \le n$. We now turn to obtaining estimates for $\partial_n \partial_n \mathbf{u}^*$. We do this by controlling it in terms of the other second partial derivatives that we already know.

Let us denote the entries of the matrix $\mathbf{J}^{-1}$ by J^{ij}. Then, the partial differential equation in the flattened coordinates reads

$$\partial_j (|\det \mathbf{J}| J^{jk} J^{lk} (\nu_1 + \nu_2 \partial_q u_m^* J^{qr} \partial_s u_m^* J^{sr}) \partial_l u_i^*) = |\det J| f_i^*.$$

By interior regularity, this equation holds almost everywhere in B_R^+. We are only interested in terms involving two derivatives with respect to y_n. So, we move all other terms to the right hand side and denote collectively the right hand side by $\mathbf{F}$. Note that all of these terms can be controlled by our previous estimates. We arrive at the equation

$$J^{nr} J^{nr} (\nu_1 + \nu_2 \partial_k u_s^* J^{kl} \partial_m u_s^* J^{ml}) + 2\nu_2 J^{ns} J^{nk} \partial_l u_j^* J^{lk} \partial_m u_i^* J^{ms} = F_i$$

Thus we are lead to consider the invertibility of the matrix

$$\delta_{ij} + \frac{2\nu_2 J^{ns} J^{nk} \partial_l u_j^* J^{lk} \partial_m u_i^* J^{ms}}{J^{nr} J^{nr} (\nu_1 + \nu_2 \partial_k u_s^* J^{kl} \partial_m u_s^* J^{ml})}$$

This symmetric matrix exists almost everywhere since $J^{nr} J^{nr} > 0$ ($\mathbf{J}$ is invertible). It is also positive definite since

$$\begin{aligned} \delta_{ij} \xi_i \xi_j + \frac{2\nu_2 J^{ns} J^{nk} \partial_l u_j^* J^{lk} \partial_m u_i^* J^{ms}}{J^{nr} J^{nr} (\nu_1 + \nu_2 \partial_k u_s^* J^{kl} \partial_m u_s^* J^{ml})} \xi_i \xi_j &= |\boldsymbol{\xi}|^2 + \frac{2\nu_2 |J^{nk} \partial_s u_j^* J^{sk} \xi_j|^2}{J^{nr} J^{nr} (\nu_1 + \nu_2 \partial_k u_s^* J^{kl} \partial_m u_s^* J^{ml})} \\ &\ge |\boldsymbol{\xi}|^2 \end{aligned}$$

Moreover, since $\left|\frac{2\nu_2 J^{ns} J^{nk}\partial_l u_j^* J^{lk}\partial_m u_i^* J^{ms}}{J^{nr}J^{nr}(\nu_1+\nu_2\partial_k u_s^* J^{kl}\partial_m u_s^* J^{ml})}\right| \leq 2$,

$$\delta_{ij}\xi_i\xi_j + \frac{2\nu_2 J^{ns} J^{nk}\partial_l u_j^* J^{lk}\partial_m u_i^* J^{ms}}{J^{nr}J^{nr}(\nu_1+\nu_2\partial_k u_s^* J^{kl}\partial_m u_s^* J^{ml})}\xi_i\xi_j \leq 3|\xi|^2.$$

We have thus established that the matrix is invertible with inverse uniformly bounded in B_R^+. Therefore

$$\begin{aligned}\int_{B_R^+} \eta^2 \nu(\nabla \mathbf{u}^*)|\partial_n\partial_n \mathbf{u}^*|^2\,d\mathbf{y} &< c(\partial\Omega)\int_{B_R^+} \frac{1}{\nu(\nabla\mathbf{u}^*)}|\mathbf{F}|^2\eta^2\,d\mathbf{y} \\ &< c(\partial\Omega,\nu_1)\left[\int_{B_R^+}\nu(\nabla u)\,|\nabla\mathbf{u}^*|^2\;d\mathbf{y} + \int_{B_R^+}|\det\mathbf{J}||\mathbf{f}|^2\,d\mathbf{y}\right].\end{aligned} \tag{3.38}$$

Combining (3.37) and (3.38) together with the interior estimate (3.20) and a simple covering estimate based on the compactness of Ω we have our final global estimate

$$||(\nu(\nabla\mathbf{u}))^{\frac{1}{2}}\nabla^2\mathbf{u}||_{L^2(\Omega)} \leq c(\Omega,\nu_1)\left[||\mathbf{f}||^2_{L^2(\Omega)} + ||(\nu(\nabla\mathbf{u}))^{\frac{1}{2}}\nabla\mathbf{u}||^2_{L^2(\Omega)}\right] \tag{3.39}$$

3.4 A Simple Application

Let us now use our L^2 regularity to prove the existence of $W^{2,2}$ solutions of the stationary vector-Burgers-like system

$$-\partial_j\left[\left(\nu_1+\nu_2|\nabla\mathbf{u}|^2\right)\partial_j u_i\right] + u_j\partial_j u_i = f_i \tag{3.40}$$

where $\mathbf{f}\in L^2$. We define the operator $\mathcal{F}_t$ from $W_0^{1,4}$ to $W_0^{1,4}$ by $\mathcal{F}_t(\mathbf{v}) = \mathbf{u}$ where $\mathbf{u}$ is the unique weak solution of

$$-\partial_j\left[\left(\nu_1+\nu_2\|\nabla\mathbf{u}|^2\right)\partial_j u_i\right] = f_i - t v_j\partial_j v_i.$$

Notice that by Sobolev's inequality, in three dimensions $\mathbf{v}$ is continuous, so $v_j\partial_j v_i \in L^2$. Thus $\mathcal{F}_t$ is well defined. From the estimate (3.3) together with the fact that

$$||\mathbf{u}-\mathbf{v}||^4_{W_0^{1,4}} < \int \nu(\nabla\mathbf{u})\nabla\mathbf{u} : \nabla(\mathbf{u}-\mathbf{v}) - \nu(\nabla\mathbf{v})\nabla\mathbf{v} : \nabla(\mathbf{u}-\mathbf{v})\,d\mathbf{x} \tag{3.41}$$

we easily obtain the continuity of this map uniformly in t from $W_0^{1,4}$ to $W_0^{1,4}$. Inequality (3.41)) follows from arguments used in Corollary 4.1 of Chapter 4. For interest and motivation, we have also proven it directly in Appendix A.

Since $\mathbf{f}$ and $v_j\partial_j v_i$ are both in L^2, the $W^{2,2}$ estimate (3.39) together with the compact embedding of $W^{2,2} \cap W_0^{1,4}$ in $W_0^{1,4}$ imply that $\mathcal{F}_t$ is compact.

Moreover, when $t = 0$, we have a unique fixed point of the map, namely the solution of the system studied in the previous sections of this Chapter. Therefore, to apply the Leray-Schauder fixed point theorem, we need only obtain an *a priori* estimate of the $W_0^{1,4}$ norm of a fixed point for each t.

However, if $\mathbf{u}$ is a fixed point then we have using $\mathbf{u}$ as a test function in the weak formulation of (3.40)

$$
\begin{aligned}
\nu_2||\nabla\mathbf{u}||_{L^4}^4 &< \int_\Omega \nu(\nabla\mathbf{u})\partial_i u_j \partial_i u_j \, d\mathbf{x} \\
&= \int_\Omega f_j u_j \, d\mathbf{x} + t\int_\Omega u_j\partial_j u_i u_i \, d\mathbf{x} \\
&\leq ||\mathbf{f}||_{L^2}||\nabla\mathbf{u}||_{L^4}|\Omega|^{\frac{1}{4}} + tc(\Omega)||\nabla\mathbf{u}||_{L^4}^3|\Omega|^{\frac{1}{4}} \\
&\leq c(\Omega,\nu_2)\left[||\mathbf{f}||_{L^2}^{\frac{4}{3}} + t^4\right] + \frac{\nu_2}{2}||\nabla\mathbf{u}||_{L^4}^4
\end{aligned}
$$

where we have used the Poincaré inequality to get this estimate.

This gives us the desired *a priori* estimate (independent of t in $[0,1]$)

$$
||\nabla\mathbf{u}||_{L^4}^4 < c(\Omega,\nu_2)\left[||\mathbf{f}||_{L^2}^{\frac{4}{3}} + 1\right]. \tag{3.42}
$$

Therefore, we may conclude by the Leray-Schauder fixed point theorem that there exists a fixed point in $W^{2,2} \cap W_0^{1,4}$ to each operator $\mathcal{F}_t$ and, in particular, there exists at least one solution in the same class to the system (3.40).

We now turn to applying the ideas of this Chapter to the Stokes-like system (2.1).

Chapter 4
Interior Regularity for the Stokes-Like Problem

We shall now concentrate on the interior regularity of weak solutions of system (1.2). As mentioned in the Introduction, we will do this in a more generalized setting. We study weak solutions of the system

$$
\begin{aligned}
-\operatorname{div} \mathbf{T}(\mathbf{D}\mathbf{u}) &= -\nabla\pi + \mathbf{f} \\
\operatorname{div} \mathbf{u} &= 0 \qquad (4.1) \\
\mathbf{u}|_{\partial\Omega} &= 0.
\end{aligned}
$$

where $\mathbf{T}$ is a C^1 function mapping $\mathbf{R}^{n\times n}_{symm}$ to $\mathbf{R}^{n\times n}_{symm}$ such that for some $p \geq 2$

$$\partial_{kl}T_{ij}(\mathbf{A})B_{ij}B_{kl} \geq c_1(1 + |\mathbf{A}|^{p-2})|\mathbf{B}|^2 \tag{4.2}$$

$$|\partial_{kl}T_{ij}(\mathbf{A})| \leq c_2(1 + |\mathbf{A}|^{p-2}) \tag{4.3}$$

for all symmetric n dimensional matrices $\mathbf{A}$ and $\mathbf{B}$. Let us first ensure that these conditions are satisfied by the Stokes-like system (2.1). In this case, $\mathbf{T}(\mathbf{A}) = (2\nu_1+2\nu_2|\mathbf{A}|^2)\mathbf{A}$, so $\partial_{kl}T_{ij}(\mathbf{A}) = (2\nu_1 + 2\nu_2|\mathbf{A}|^2)\delta_{ik}\delta_{jl} + 4\nu_2 A_{kl}A_{ij}$. Therefore

$$
\begin{aligned}
\partial_{kl}T_{ij}(\mathbf{A})B_{kl}B_{ij} &= (2\nu_1 + 2\nu_2|\mathbf{A}|^2)|\mathbf{B}|^2 + 4\nu_2 A_{kl}B_{kl}A_{ij}B_{ij} \qquad (4.4) \\
&\geq \min(2\nu_1, 2\nu_2)(1 + |\mathbf{A}|^2)|\mathbf{B}|^2 \qquad (4.5)
\end{aligned}
$$

and

$$|\partial_{kl}\mathbf{T}_{ij}(\mathbf{A})| = |(2\nu_1 + 2\nu_2|\mathbf{A}|^2)|\mathbf{B}|^2 + 4\nu_2 A_{kl}A_{ij}| \tag{4.6}$$

$$\leq \max(2\nu_1, 6\nu_2)(1 + |\mathbf{A}|^2). \tag{4.7}$$

Thus our stress tensor is of the type considered. Indeed, the conditions (4.2) and (4.3) are essentially those of ellipticity with a natural growth condition.

Let $\mathbf{f}$ be in $(W_0^{1,p})^*$. Then we say that $\mathbf{u}$ is a weak solution of (1.2) if is in $W_0^{1,p}$ and satisfies

$$\int_\Omega T_{ij}(\mathbf{Du})D_{ij}\phi\, d\mathbf{x} =< \mathbf{f}, \phi > \tag{4.8}$$

for all ϕ in $\mathcal{J}$ (and therefore by continuity for all ϕ in $W_0^{1,p}(\Omega)$). From the properties (4.2) and (4.3) of $\mathbf{T}$ together with the Fundamental Theorem of Calculus it is well known [MNRR96] that $\mathbf{T}$ also satisfies

$$T_{ij}(\mathbf{A}) \cdot A_{ij} \geq c(c_1, p)(1 + |\mathbf{A}|^{p-2})|\mathbf{A}|^2, \tag{4.9}$$

$$|T_{ij}(\mathbf{A})| \leq c(c_2, n)(1 + |\mathbf{A}|^{p-2})|\mathbf{A}|, \tag{4.10}$$

$$(T_{ij}(\mathbf{A}) - T_{ij}(\mathbf{B})) \cdot (\mathbf{A} - \mathbf{B})_{ij} \geq c_1|\mathbf{A} - \mathbf{B}|^2. \tag{4.11}$$

From these properties, it is not difficult to show, for example, with standard Galerkin techniques and monotone operator theory that weak solutions of (1.2) exist. Moreover, from property (4.11) we see that weak solutions are unique. These results (although not always the methods used to prove them) are analogous to those determined for the Stokes-like system in Chapter 2. Aside for these cursory comments on existence we ignore this topic for the remainder of this thesis keeping in mind its scope and hoping to focus on the regularity issues.

We now consider the central difficulty presented by the regularity theory of equation (4.8), namely that the weak pressure is *a priori* in a less regular space than $L^2(\Omega)$, as seen for the Stokes-like system in Chapter 2. Indeed, if f is in $L^2(\Omega)$, the functional $\mathcal{L}$

defined on $\mathbf{v}$ in $W_0^{1,p}$ by

$$\mathcal{L}(\mathbf{v}) = \int_\Omega T_{ij}(\mathbf{Du})D_{ij}(\mathbf{v}) - \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}$$

is linear and bounded on $W_0^{1,p}$ and vanishes on $J_0^{1,p}$, so it follows from standard Navier-Stokes theory [Gal94] that there exists a function π in $L_{loc}^{p'}(\Omega)$ such that

$$\int_\Omega T_{ij}(\mathbf{Du})D_{ij}(\phi) - \mathbf{f}\cdot\phi\,d\mathbf{x} = \int_\Omega \pi\,\mathrm{div}\,\phi\,d\mathbf{x} \tag{4.12}$$

for all ϕ in $\mathcal{J}$. Here lies the problem. Although the $W_0^{1,p}$ weak solution $\mathbf{u}$ has an initially known higher degree of regularity than, for example, $W_0^{1,2}$ weak solutions of the Stokes system, the corresponding pressure arising from basic Navier-Stokes theory potentially lies in a less regular space: $L^{p'}$ instead of L^2. Proceeding formally, if η is a cut off function corresponding to interior regularity and we use $\eta^2\partial_i\partial_i\mathbf{u}$ as a test function in (4.12), as we did for the *a priori* estimate of Chapter 3, we must bound from above a term of the form

$$\int_\Omega \pi\eta\left(\nabla\eta\cdot\partial_i\partial_i\mathbf{u}\right)\,d\mathbf{x}.$$

in terms of the L^2 norm of the second derivatives of $\mathbf{u}$. Just applying Hölder's inequality would then create a integral such as

$$\int_\Omega |\nabla\eta|^2\pi^2\,d\mathbf{x}$$

which is not controlled by knowing π is in $L_{loc}^{p'}(\Omega)$. This problem does not arise in the case of periodic boundary conditions since $\Delta\mathbf{u}$ is a solenoidal test function. We only encounter this problem when we attempt to localize our test function, which is necessary for Dirichlet boundary conditions.

Hence there are two obvious lines of attack. Either use the fact that $\mathbf{f}$ is more regular than just being in $(W_0^{1,p})^*$ to prove something stronger about the pressure or find a solenoidal test function that avoids dealing with the pressure. The approach for the system (4.1) has been to prove, essentially, that the pressure lies in $L^2(\Omega)$ [MNR96]. To do this, however, one assumes that $\mathbf{T}$ is derived from a potential (so $T_{ij} = \partial_{ij} F(|\mathbf{Du}|^2)$ for some scalar valued function F). The proof then continues by approximating the potentials from which $\mathbf{T}$ is derived with potentials exhibiting linear growth in $|\mathbf{Du}|^2$ at infinity and then taking limits. Until now, a solenoidal test function approach has not been used and in doing so we eliminate an assumption on the form of $\mathbf{T}$, although at the cost of not yet being able to prove boundary regularity.

To motivate our method, let us recall the solenoidal test function approach to interior regularity for the Stokes operator. We will work formally assuming that $\mathbf{u}$ is smooth. Following [Lad69] we let η be a cut-off function corresponding to interior regularity and use the the fact that the curl of a vector field is solenoidal to set $\phi = \operatorname{curl}(\eta^2 \operatorname{curl} \mathbf{u})$ in (4.8). Then, distributing the curl operation in ϕ, we see

$$\begin{aligned} \phi &= \eta^2(-\Delta \mathbf{u} + \nabla(\operatorname{div} \mathbf{u})) + \nabla \eta^2 \times \operatorname{curl} \mathbf{u} \\ &= -\eta^2 \Delta \mathbf{u} + 2\eta \left(\nabla \eta \times \operatorname{curl} \mathbf{u}\right). \end{aligned} \tag{4.13}$$

For the Stokes system, $T_{ij}(\mathbf{Du}) = D_{ij}(\mathbf{u})$ and we quickly arrive after some integrations by parts and applications of Hölder's inequality to the desired estimate

$$\int_\Omega \eta^2 |\nabla^2 \mathbf{u}|^2 \, d\mathbf{x} \le c \left[\int_\Omega \eta^2 |\mathbf{f}|^2 + |\nabla \eta|^2 |\nabla \mathbf{u}|^2 \, d\mathbf{x} \right]. \tag{4.14}$$

Perhaps not surprisingly, the same test-function ϕ yields a similar *a priori* estimate for the general case. This time, we will be more explicit since the *a priori* estimate so obtained will be the sharpest estimate we shall see and should be the goal of our later calculations. For our test function we set

$$\phi_i = -\eta^2 \Delta u_i + \partial_j \eta^2 \partial_i u_j - \partial_j \eta^2 \partial_j u_i \tag{4.15}$$

which is easily seen to correspond with (4.13) in the three dimensional case and avoids having to reinterpret the curl and cross-product in higher dimensional settings. Indeed, Heywood used this variant of (4.13) in [Hey76] in discussing interior regularity of the Stokes operator. Again, we assume $\mathbf{u}$ to be smooth in the following computations. Using (4.15) as the test function in (4.8) and integrating by parts we get

$$\begin{aligned}\int_\Omega \eta^2 \partial_j T_{kl}(\mathbf{Du}) \partial_k \partial_j u_l \, d\mathbf{x} &= \int_\Omega \partial_k (T_{kl}(\mathbf{Du})) \partial_j (\eta^2 \partial_l u_j) - \\ &\qquad - \partial_j (T_{kl}(\mathbf{Du})) \partial_k \eta^2 \partial_j u_l + T_{ij}(\mathbf{Du}) \partial_j \phi_i \, d\mathbf{x} \\ &= \int_\Omega \partial_k (T_{kl}(\mathbf{Du})) \partial_j (\eta^2 \partial_l u_j) - \partial_j (T_{kl}(\mathbf{Du})) \partial_k \eta^2 \partial_j u_l + f_i \phi_i \, d\mathbf{x}.\end{aligned} \tag{4.16}$$

From the estimate (4.2) we obtain immediately

$$\begin{aligned}\int_\Omega \eta^2 \partial_j T_{kl}(\mathbf{Du}) \partial_k \partial_j u_l \, d\mathbf{x} &= \int_\Omega \eta^2 \partial_{mn} \left(T_{kl}(\mathbf{Du}) \right) D_{mn}(\partial_j \mathbf{u}) D_{kl}(\partial_j \mathbf{u}) \, d\mathbf{x} \\ &\geq c_1 \sum_j \int_\Omega \eta^2 (1 + |\mathbf{Du}|^{p-2}) |\partial_j \mathbf{Du}|^2 \, d\mathbf{x}.\end{aligned} \tag{4.17}$$

Now we need to estimate from above the three integrals on the right hand side of (4.16). For the first we have using (4.3) and Hölder's inequality,

$$\begin{aligned}\left| \int_\Omega \partial_k T_{kl} \partial_j \eta^2 \partial_l u_j \, d\mathbf{x} \right| &= \left| \int_\Omega \partial_{mn} \left(T_{kl}(\mathbf{Du}) \right) D_{mn}(\partial_k u) 2\eta \partial_j \eta \partial_l u_j \right| \, d\mathbf{x} \\ &\leq \sum_k \int_\Omega c(c_2, n)(1 + |\mathbf{Du}|^{p-2}) \eta |\partial_k \mathbf{Du}| |\nabla \eta| |\mathbf{Du}| \, d\mathbf{x} \\ &\leq \sum_j \int_\Omega c(c_2, n)(1 + |\mathbf{Du}|^{p-2}) \eta |\partial_j \mathbf{Du}| |\nabla \eta| |\mathbf{Du}| \, d\mathbf{x} \\ &\leq \int_\Omega c(c_1, c_2, n)(1 + |\mathbf{Du}|^{p-2}) |\nabla \eta|^2 |\mathbf{Du}|^2 + \\ &\qquad + \epsilon(c_1) \sum_j \int_\Omega c(1 + |\mathbf{Du}|^{p-2}) \eta^2 |\partial_j \mathbf{Du}|^2 \, d\mathbf{x}.\end{aligned} \tag{4.18}$$

For the second we have the similar estimate, also using (4.3) and Hölder's inequality,

$$\begin{aligned}\left|\int_\Omega \partial_j T_{kl}\partial_k\eta^2\partial_j u_l\,d\mathbf{x}\right| &\leq \int_\Omega |\partial_{mn}T_{kl}(\mathbf{Du})||D_{mn}\partial_j u|2\eta|\partial_k\eta||\partial_j u_l|\,d\mathbf{x}\\ &\leq \sum_j \int_\Omega c(c_2,n)(1+|\mathbf{Du}|^{p-2})|\partial_j\mathbf{Du}|\eta|\nabla\eta||\mathbf{Du}|\,d\mathbf{x}\\ &\leq \int_\Omega c(c_1,c_2,n)(1+|\mathbf{Du}|^{p-2})|\nabla\eta|^2|\mathbf{Du}|^2\,d\mathbf{x}+\\ &\qquad+\sum_j \epsilon(c_1)\int_\Omega \eta^2(1+|\mathbf{Du}|^{p-2})|\partial_j\mathbf{Du}|^2\,d\mathbf{x}. \end{aligned}\tag{4.19}$$

For the last estimate we have by simple application of Hölder's inequality,

$$\begin{aligned}\left|\int_\Omega f_i\phi_i\,d\mathbf{x}\right| &= \left|\int_\Omega -\eta^2 f_i\Delta u_i + f_i\partial_j\eta^2(\partial_i u_j - \partial_j u_i)\,d\mathbf{x}\right|\\ &\leq c(c_1)\int_\Omega \left(\eta^2|\mathbf{f}|^2 + |\nabla\eta|^2|\mathbf{Du}|^2\right)\,d\mathbf{x} + \epsilon(c_1)\int_\Omega \eta^2|\Delta u|^2\,d\mathbf{x}. \end{aligned}\tag{4.20}$$

We would like to bound the last term in (4.20) in terms of derivatives of $\mathbf{Du}$ so as to incorporate it into the left hand side. To do this, we note that for smooth solenoidal functions $\mathbf{v}$,

$$\begin{aligned}\int_\Omega \eta^2|\mathbf{Dv}|^2\,d\mathbf{x} &= \frac{1}{2}\int_\Omega \eta^2(\partial_i v_j\partial_i v_j + \partial_j v_i\partial_i v_j)\,d\mathbf{x}\\ &\geq \int_\Omega \frac{1}{4}\eta^2|\nabla\mathbf{v}|^2 - 4|\nabla\eta|^2|\mathbf{v}|^2\,d\mathbf{x}. \end{aligned}$$

Thus,

$$\int_\Omega \eta^2|\nabla\mathbf{v}|^2\,d\mathbf{x} \leq 16\int_\Omega |\nabla\eta|^2|\mathbf{v}|^2 + \eta^2|\mathbf{Dv}|^2\,d\mathbf{x}. \tag{4.21}$$

Since smooth solenoidal functions are dense in $J_0^{1,2}$ and since the integrals in (4.21) are continuous on this space, the result holds for all $\mathbf{v}$ in $J_0^{1,2}$, and we will have occasion to use this fact later.

Thus, letting $\mathbf{v} = \partial_j\mathbf{u}$ in (4.21) and summing on j we find that for smooth solutions,

$$\begin{aligned}&\sum_j \int_\Omega \eta^2(1+|\mathbf{Du}|^{p-2})|\partial_j\mathbf{Du}|^2\,d\mathbf{x} \leq\\ &\qquad c(c1)\int_\Omega \eta^2|\mathbf{f}|^2\,d\mathbf{x} + c(c1,c2,n)\int_\Omega |\nabla\eta|^2\left[|\nabla\mathbf{u}|^2 + |\mathbf{Du}|^p\right]\,d\mathbf{x}. \end{aligned}\tag{4.22}$$

Our goal now is to find a way to incorporate these ideas into a proof. To do this, we will first need to take care of some preliminaries.

4.1 Preliminary Lemmas

To begin, we consider the smearing operator σ defined in the introduction. We now prove a simple lemma showing that its action on Sobolev spaces is well behaved.

Lemma 4.1 *Let $v \in W^{1,p}(\Omega)$ and let Ω' be an open set with closure contained in Ω with $d(\Omega', \partial\Omega) > h$. Then the function*

$$\sigma_{h,m} v(\mathbf{x}) = \int_0^1 v(\mathbf{x} + th\mathbf{e}_m)\, dt$$

is in $W^{1,p}(\Omega')$ with

$$\partial_i \sigma_{h,m} v(\mathbf{x}) = \int_0^1 \partial_i v(\mathbf{x} + th\mathbf{e}_m)\, dt. \tag{4.23}$$

Moreover,

$$||\sigma_{h,m}\mathbf{v}||_{W^{1,p}(\Omega')} \leq ||\mathbf{v}||_{W^{1,p}(\Omega'')} \tag{4.24}$$

where Ω'' is any open set such that $\Omega \supset \Omega'' \supset \Omega'$ and $d(\Omega', \partial\Omega'') > h$.

Proof:

Let us first show that $\sigma_{h,m} v$ has generalized first derivatives. If ψ is in $C_0^\infty(\Omega')$, then

$$\int_\Omega (\sigma_{h,m} v)\, \partial_i \psi\, d\mathbf{x} = \int_\Omega \int_0^1 v(\mathbf{x} + th\mathbf{e}_m) \partial_i \psi\, dt\, d\mathbf{x}.$$

By Fubini's theorem, together with the fact that $v(\mathbf{x} + th\mathbf{e}_m)$ is bounded uniformly in t in $W^{1,p}(\Omega')$, we are able to change the order of integration. We also change variables to get

$$\int_\Omega (\sigma_{h,m} v)\, \partial_i \psi\, d\mathbf{x} = \int_0^1 \int_\Omega v(\mathbf{y}) \partial_i \psi(\mathbf{y} - th\mathbf{e}_m)\, d\mathbf{y}\, dt.$$

Now we use Fubini again to change the order of integration and use the fact that ψ is smooth to exchange the order of integration and differentiation to obtain

$$\int_{\Omega} (\sigma_{h,m} v)\, \partial_i \psi \, d\mathbf{x} = \int_{\Omega} v(\mathbf{y}) \partial_i \int_0^1 \psi(\mathbf{y} - th\mathbf{e}_m)\, dt\, d\mathbf{y}.$$

Since v has generalized derivatives, we can integrate by parts and change variables a final time to arrive at the desired equation

$$\int_{\Omega'} (\sigma_{h,m} v)\, \partial_i \psi \, d\mathbf{x} = -\int_{\Omega'} \int_0^1 \partial_i v(\mathbf{x} + th\mathbf{e}_m)\, dt \psi \, d\mathbf{x}.$$

Since ψ is arbitrary in $C_0^\infty(\Omega')$ we conclude that $\partial_i \sigma_{h,m} v$ exists and is $\int_0^1 \partial_i v(\mathbf{x}+th\mathbf{e}_m)\, dt$.

To show that $\partial_i \sigma_{h,m} v$ is in L^p we use Jensen's inequality and Fubini. It follows that

$$\begin{aligned} \int_{\Omega'} |\partial_i \sigma_{h,m} v|^p \, d\mathbf{x} &\leq \int_{\Omega'} \int_0^1 |\partial_i v(\mathbf{x} + th\mathbf{e}_m)|^p \, dt\, d\mathbf{x} \\ &\leq \int_0^1 \int_{\Omega'} |\partial_i v(\mathbf{x} + th\mathbf{e}_m)|^p \, d\mathbf{x} dt \\ &\leq \int_{\Omega''} |\partial_i v(x)|^p d\Omega, \end{aligned}$$

where Ω'' is any set satisfying the properties assumed.

□

Our regularity proof will use as before method of difference quotients. In the course of our calculations, we will have occasion to estimate from below the integral

$$\int_0^1 |t\mathbf{A} + (1-t)\mathbf{B}|^q \; dt.$$

This useful estimate is claimed but not proven in [LU68]. We present its proof here since it has a nice, geometrically motivated, argument and gives us the opportunity to put a lone figure in this thesis. If we are looking for a lower bound for

$$\int_0^1 |t\mathbf{A} + (1-t)\mathbf{B}|^q \; dt.$$

we can equate this integral with an average value of the function $|y|^q$ taken along the line from $\mathbf{A}$ to $\mathbf{B}$. Given a fixed $\mathbf{B}$, it seems reasonable the lowest path integral of all

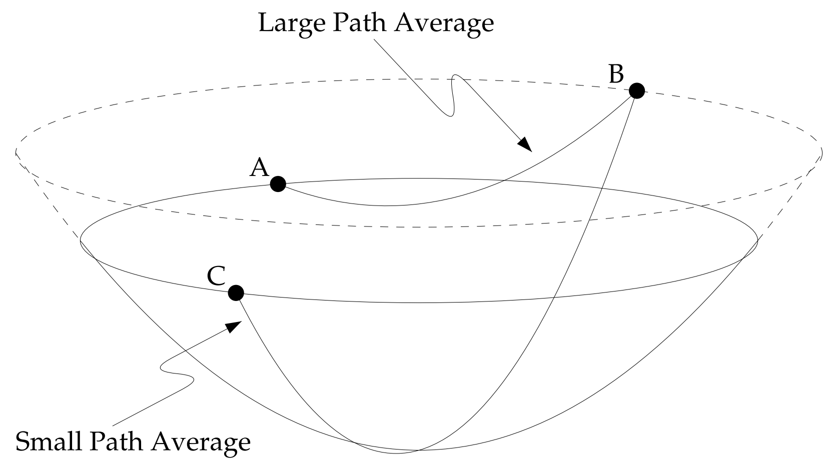

Figure 4.1: Comparison of Path Integrals

points $\mathbf{C}$ on the same level set as $\mathbf{A}$ is the one that passes through 0 (see Figure 4.1). Since this point is on the line that connects $\mathbf{B}$ and 0, the argument reduces to the one dimensional case, which is easy to prove.

Lemma 4.2 *Let $\mathbf{A}$ and $\mathbf{B}$ be any two matrices in $\mathbf{R}^{n\times n}$. Then for all $q \geq 0$,*

$$\int_0^1 |t\mathbf{A} + (1-t)\mathbf{B}|^q \; dt \geq c(q)\left(|\mathbf{A}|^q + |\mathbf{B}|^q\right). \tag{4.25}$$

Proof:

Since the integral on the left hand side of (4.25) is symmetric in $\mathbf{A}$ and $\mathbf{B}$ by change of variables , we may assume without loss of generality that $|\mathbf{B}| = \delta|\mathbf{A}|$ for some δ in $[0,1]$. Since

$$\begin{aligned} |t\mathbf{A} + (1-t)\mathbf{B}| &\geq |(|t\mathbf{A}| - |(1-t)\mathbf{B}|)| \\ &= |(t\delta|\mathbf{B}| - (1-t)|\mathbf{B}|)| \\ &= |t(1+\delta) - 1||\mathbf{B}| \end{aligned}$$

for all t in $[0,1]$, and since y^q is non-decreasing for all $q \geq 0$ and $y \geq 0$, it follows that

$$|t\mathbf{A} + (1-t)\mathbf{B}|^q \geq |t(1+\delta) - 1|^q |\mathbf{B}|^q.$$

Integrating this expression in t yields

$$\begin{aligned} \int_0^1 |t\mathbf{A} + (1-t)\mathbf{B}|^q \, dt &\geq \int_0^1 |t(1+\delta) - 1|^q |\mathbf{B}|^q \, dt \\ &= |\mathbf{B}|^q \frac{1}{1+\delta} \int_0^{1+\delta} |1-w|^q \, dw \\ &\geq |\mathbf{B}|^q \frac{1}{2} \int_0^1 (1-w)^q \, dw \\ &\geq \frac{1}{4(1+q)} \left(|\mathbf{A}|^q + |\mathbf{B}|^q \right), \end{aligned}$$

which proves estimate (4.25) with $c(q) = 1/(4(q+1))$.

□

A Corollary to Lemma 4.2 is an improvement of estimate (4.11). We will use this improvement in our study of the regularity of steady solutions of (1.1).

Corollory 4.1 *Let* $\mathbf{T}$ *satisfy (4.2) for some* $p \geq 2$. *Then*

$$(T_{ij}(\mathbf{A}) - T_{ij}(\mathbf{B}))(\mathbf{A} - \mathbf{B})_{ij} \geq c_1 |\mathbf{A} - \mathbf{B}|^2 + c(c_1, p) |\mathbf{A} - \mathbf{B}|^p. \tag{4.26}$$

Proof:

From the Fundamental Theorem of Calculus and (4.2) it is easy to see that

$$\begin{aligned} (T_{ij}(\mathbf{A}) - T_{ij}(\mathbf{B}))(\mathbf{A} - \mathbf{B})_{ij} &= \int_0^1 \partial_{rs} T_{ij}(t\mathbf{A} + (1-t)\mathbf{B})(\mathbf{A} - \mathbf{B})_{rs} (\mathbf{A} - \mathbf{B})_{ij} \, dt \\ &\geq \int_0^1 c_1 \left(1 + |t\mathbf{A} + (1-t)\mathbf{B}|^{p-2}\right) \, dt |\mathbf{A} - \mathbf{B}|^2. \end{aligned}$$

From Lemma (4.2) we conclude

$$\begin{aligned} (T_{ij}(\mathbf{A}) - T_{ij}(\mathbf{B}))(\mathbf{A} - \mathbf{B})_{ij} &\geq c_1 |\mathbf{A} - \mathbf{B}|^2 + c(c_1, p) \left(|\mathbf{A}|^{p-2} + |\mathbf{B}|^{p-2} \right) |\mathbf{A} - \mathbf{B}|^2 \\ &\geq c_1 |\mathbf{A} - \mathbf{B}|^2 + c(c_1, p) |\mathbf{A} - \mathbf{B}|^p. \end{aligned}$$

□

The primary application of Lemma 4.2 is to take advantage of the ellipticity property (4.2) of $\mathbf{T}$ in a difference quotient setting. In our final preliminary Lemma we prove this classical [LU68] coercivity estimate together with a growth estimate. These are analogous to the properties proven for the bilinear form $\mathbf{a}^{h,m}$ of Chapter 3.

Lemma 4.3 *Let $\mathbf{T}$ satisfy the ellipticity property (4.2) and the growth property (4.3) for some $p \geq 2$. Then for every $\mathbf{u}$ in $J_0^{1,p}(\Omega)$ and every $\mathbf{B}$ in $\mathbf{R}^{n\times n}_{symm}$ and for almost every $\mathbf{x}$ in Ω,*

$$\tau_{h,s}(T_{ij}(\mathbf{Du}(\mathbf{x})))\tau_{h,s}D_{ij}\mathbf{u} \geq c(c_1,p)(1+|\mathbf{Du}|^{p-2}+|\mathbf{Du}|^{p-2}_{\mathbf{x}+h\mathbf{e}_s})|\tau_{h,s}\mathbf{Du}|^2 \qquad (4.27)$$

and

$$|\tau_{h,s}(T_{ij}(\mathbf{Du}(\mathbf{x})))B_{ij}| \leq c(c_2,n,p)(1+|\mathbf{Du}|^{p-2}_{\mathbf{x}+h\mathbf{e}_s}+|\mathbf{Du}|^{p-2}|)|\tau_{h,s}\mathbf{Du}||\mathbf{B}|. \qquad (4.28)$$

Proof:

For the bound (4.27) we use the fact that for almost every $\mathbf{x}$,

$$\begin{aligned}\tau_{h,s}(T_{ij}(\mathbf{Du}(\mathbf{x})) &= \frac{1}{h}\int_0^1 \frac{d}{dt}T_{ij}(th\tau_{h,s}(\mathbf{Du}(\mathbf{x}))+\mathbf{Du}(\mathbf{x}))\,dt\\ &= \int_0^1 \partial_{kl}T_{ij}(th\tau_{h,s}(\mathbf{Du}(\mathbf{x}))\tau_{h,s}D_{kl}\mathbf{u}\,dt. \qquad (4.29)\end{aligned}$$

Hence, contracting (4.29) with $\tau_{h,s}D_{ij}\mathbf{u}$ we obtain from (4.2)

$$\tau_{h,s}(T_{ij}(\mathbf{Du}(x)) \geq \int_0^1 c_1(1+\left|(t\,\mathbf{Du}|_{\mathbf{x}+h\mathbf{e}_s}+(1-t)\mathbf{Du})\right|^{p-2})\,dt|\tau_{h,s}\mathbf{Du}|^2.$$

From inequality (4.25) of Lemma 4.2 we obtain immediately (4.27).

For the second estimate (4.28) we use the growth assumption (4.3) together with the property that y^{p-1} is convex for every $p \geq 2$ to find that for almost every $\mathbf{x}$

$$
\begin{aligned}
|\tau_{h,s}(T_{ij}(\mathbf{Du}))B_{ij}| &= \left| \int_0^1 \partial_{kl} T_{ij}(th\tau_{h,s}\mathbf{Du} + \mathbf{Du}) D_{kl}(\tau_{h,s}\mathbf{u})\, dt B_{ij} \right| \\
&\leq c(c_2, n) \int_0^1 (1 + \left| \left(t\ \mathbf{Du}|_{x+he_s} + (1-t)\mathbf{Du} \right) \right|^{p-2})\, dt\, |\tau_{h,s}\mathbf{Du}||\mathbf{B}| \\
&\leq c(c_2, n, p) \left(1 + |\mathbf{Du}|^{p-2}_{\mathbf{x}+h\mathbf{e}_s} + |\mathbf{Du}|^{p-2} \right) |\tau_{h,s}\mathbf{Du}||\mathbf{B}|.
\end{aligned}
$$

□

4.2 A New Test Function

Our next goal is to find a solenoidal test function to use in the difference quotient context to arrive at interior regularity. We are motivated by the form of the test function (4.15). Therefore, let us write

$$
\phi_i = -\eta^2 \sum_j \tau_{-h,j}\tau_{h,j}u_i + \partial_j \eta^2 \Psi_{ij} - \partial_j \eta^2 \Psi_{ji} \tag{4.30}
$$

where Ψ is an unknown tensor to be determined by the solenoidal constraint. Specifically, Ψ must satisfy

$$
-\partial_i \eta^2 \sum_j \tau_{-h,j}\tau_{h,j}u_i + \partial_i \eta^2 \partial_j \Psi_{ji} - \partial_i \eta^2 \partial_j \Psi_{ij} = 0.
$$

This could be solved, as in the test function (4.15), if one could exhibit a tensor Ψ such that $\partial_j \Psi_{ij} = 0$ and $\partial_j \Psi_{ji} = \sum_j \tau_{-h,j}\tau_{h,j}u_i$. However, in the support of η, we can write the difference quotient "Laplacian" as

$$\begin{aligned}
\sum_j \tau_{-h,j}\tau_{h,j}u_i &= \sum_j \tau_{-h,j}\frac{1}{h}\int_0^1 \frac{\partial}{\partial t}u_i(\mathbf{x}+ht\mathbf{e}_j)\,dt \\
&= \sum_j \tau_{-h,j}\int_0^1 \partial_j u_i(\mathbf{x}+ht\mathbf{e}_j)\,dt \\
&= \sum_j \tau_{-h,j}\partial_j \int_0^1 u_i(\mathbf{x}+ht\mathbf{e}_j)\,dt \\
&= \sum_j \partial_j \tau_{-h,j}\sigma_{h,j}u_i \qquad (4.31)
\end{aligned}$$

where used Lemma 4.1 applied to the $W^{1,p}$ functions u_i. We define

$$\Psi_{ji} = \tau_{-h,j}\int_0^1 u_i(x+hte_j)dt = \tau_{-h,j}\sigma_{h,j}u_i. \qquad (4.32)$$

Then, applying Lemma 4.1 again and using the fact that $\mathbf{u}$ is solenoidal, we get $\partial_i\Psi_{ji} = 0$. Therefore Ψ is the desired tensor and

$$\phi_i = \sum_j \left[-\eta^2\tau_{-h,j}\tau_{h,j}u_i + \partial_j\eta^2\tau_{-h,i}\sigma_{h,i}u_j - \partial_j\eta^2\tau_{-h,j}\sigma_{h,j}u_i\right] \qquad (4.33)$$

is the test function we seek.

4.3 Interior Regularity

We now use the test function found in the previous section to prove our central result.

Theorem 4.1 *Let Ω be a bounded domain and $\mathbf{u}$ be a weak solution of (1.2) where $\mathbf{f}$ is in $(W_0^{1,p}(\Omega))^* \cap L^2_{loc}(\Omega)$ and $\mathbf{T}$ satisfies properties (4.2) and (4.3). Then $\mathbf{u}$ is in $W^{2,2}_{loc}(\Omega)$ and satisfies*

$$\int_\Omega \eta^2\left[|\nabla^2\mathbf{u}|^2 + |\mathbf{Du}|^{p-2}\sum_j|\partial_j\mathbf{Du}|^2\right]d\mathbf{x} \le c(c_1,c_2,n)\left(\int_\Omega \eta^2|\mathbf{f}|^2\,d\mathbf{x} + \int_\Omega |\nabla\eta|^2\left[|\nabla\mathbf{u}|^2 + |\mathbf{Du}|^p\right]\,d\mathbf{x}\right) \qquad (4.34)$$

for every smooth interior cut-off function η.

Proof:

We divide the proof into two parts. First we show that $\mathbf{u}$ has second derivatives. However, we will be constrained in our use of integration by parts when we show the second derivatives exist and will not obtain the sharper estimate (4.34). In the second part we use the existence of second derivatives to obtain (4.34).

Part 1. Introducing the test function ϕ defined by (4.33) into the weak formulation (4.8) we obtain

$$\begin{aligned}\sum_j \int_\Omega \eta^2 \tau_{h,j} T_{kl}(\mathbf{Du}) \tau_{h,j} D_{kl}\mathbf{u}\, d\mathbf{x} &= \int_\Omega \Big(-\sum_j T_{kl}(\mathbf{Du})|_{\mathbf{x}+h\mathbf{e}_j}\, \tau_{h,j}(\eta^2)\tau_{h,j} D_{kl}\mathbf{u} \\ &\quad + \sum_j T_{kl}(\mathbf{Du})\partial_k(\eta^2)\tau_{-h,j}\tau_{h,j} u_l \\ &\quad + T_{kl}(\mathbf{Du})\partial_k(\partial_j \eta^2 (\Psi_{ji} - \Psi_{ij})) \\ &\quad + f_i \phi_i\Big)\, d\mathbf{x} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \end{aligned} \tag{4.35}$$

where we have used the standard properties of difference quotients that can be found in [Gia93] to "integrate by parts" and to distribute a difference quotient over a product. We will bound the left hand side of (4.35) from below and the right hand side from above to get an estimate for the difference quotients of $\mathbf{Du}$. At this point it might seem sufficient to simply cite our work from Chapter 3 to claim the final estimate. However, many calculations are different from those that appear in the standard theory. We have introduced new terms in our test function of the type $\tau_{-h,j}\sigma_{h,j}\mathbf{u}$ and we must be careful to introduce approximate second derivatives of the form $\tau_{h,k}\mathbf{Du}$ only and not more general terms $\tau_{h,k}\partial_i\mathbf{u}$. Therefore, let us find bounds for (4.35) from above and below. Since we will encounter the weight $(1 + |\mathbf{Du}|^{p-2} + |\mathbf{Du}|^{p-2}_{\mathbf{x}+h\mathbf{e}_j})$ several times, we will denote it as $\mu_{h,j}$. Of course, $\mu_{h,j}$ converges strongly in $L^{\frac{p}{p-2}}$ as h goes to 0 to $(1 + 2|\mathbf{Du}|^{p-2})$ which we will simply call μ. It will also be convenient to use Ω_h to denote

$$\left(\bigcup_j \operatorname{supp}(\tau_{h,j}\eta)\right)\bigcup \operatorname{supp}(\eta)$$

which approximates $\operatorname{supp}(\eta)$ in the limit as h goes to 0.

We use the estimate (4.25) from Lemma 4.2 to bound the left hand side of (4.35) from below by

$$\begin{aligned}\sum_j \int_\Omega \eta^2 \tau_{h,j} T_{kl}(\mathbf{Du})\tau_{h,j} D_{kl}\mathbf{u}\,d\mathbf{x} &\geq c(c_1,p)\sum_j \int_\Omega \eta^2 (1+|\mathbf{Du}|^{p-2}+|\mathbf{Du}|^{p-2}_{\mathbf{x}+h\mathbf{e}_j})|\tau_{h,j}\mathbf{Du}|^2\,d\mathbf{x} \\ &= c(c_1,p)\sum_j \int_\Omega \mu_{h,j}|\tau_{h,j}\mathbf{Du}|^2\,d\mathbf{x}. \qquad (4.36)\end{aligned}$$

We now turn to maximizing each of the integrals on the right-hand side of (4.35). Using the bound (4.10) for $\mathbf{T}$ we see

$$\begin{aligned}|\mathrm{I}| &= \left|\int_\Omega \sum_j T_{kl}(\mathbf{Du})|_{\mathbf{x}+h\mathbf{e}_j}\,\tau_{h,j}(\eta^2)\tau_{h,j}D_{kl}\mathbf{u}\,d\mathbf{x}\right| \\ &\leq \sum_j c(c_2,n)\int_\Omega (1+|\mathbf{Du}|^{p-2}_{x+he_j})|\mathbf{Du}|_{x+he_j}|\tau_{h,j}(\eta^2)||\tau_{h,j}\mathbf{Du}|\,d\mathbf{x} \\ &\leq \sum_j c(c_2,n)\int_\Omega \mu_{h,j}|\mathbf{Du}|_{x+he_j}|\tau_{h,j}(\eta^2)||\tau_{h,j}\mathbf{Du}|\,d\mathbf{x}. \qquad (4.37)\end{aligned}$$

From the “Leibniz-rule” for different quotients,

$$\tau_{h,j}\eta^2 = \tau_{h,j}\eta\left(\eta + \eta|_{x+he_j}\right).$$

Since η is smooth we see $\eta|_{x+he_j} = \eta(x) + \int_0^h \frac{d}{dt}\eta(\mathbf{x}+t\mathbf{e}_j)\,dt \leq \eta(x) + hc(\nabla\eta)$ and therefore

$$|\tau_{h,j}(\eta^2)| \leq c(\nabla\eta)(\eta + h). \qquad (4.38)$$

On the other hand,

$$h|\tau_{h,j}\mathbf{Du}| \leq |\mathbf{Du}| + |\mathbf{Du}|_{\mathbf{x}+h\mathbf{e}_j}. \qquad (4.39)$$

We combine (4.38) and (4.39) in (4.37) and apply Young's inequality to get

$$|\mathrm{I}| \leq c(c_1, c_2, p, n, \nabla\eta) \sum_j \int_{\Omega_h} \mu_{h,j}(|\mathbf{Du}|^2 + |\mathbf{Du}|^2_{\mathbf{x}+h\mathbf{e}_j})\, d\mathbf{x} + \\ + \epsilon(c_1, p) \sum_j \int_\Omega \eta^2 \mu_{h,j} |\tau_{h,j}\mathbf{Du}|^2\, d\mathbf{x} \quad (4.40)$$

where we have written explicitly the dependence of ϵ on the parameters appearing in the constant in (4.36) to make clear that ϵ need only be small enough to merge into this term, along with other terms of this type appearing in the later computations.

To estimate the second integral, we must "integrate by parts" with one of the difference quotients. Doing this we have, employing the bound (4.28) of Lemma (4.3) along with (4.10) and Young's inequality,

$$\begin{aligned}
|\mathrm{II}| &= \left| \sum_j \int_\Omega \tau_{h,j}(T_{kl}(\mathbf{Du})\partial_k\eta^2)\tau_{h,j}u_l\, d\mathbf{x} \right| \\
&= \left| \sum_j \int_\Omega \left[\tau_{h,j}(T_{kl}(\mathbf{Du}))\partial_k\eta^2 + T_{kl}(\mathbf{Du})|_{x+he_j}\, \tau_{h,j}\partial_k(\eta^2) \right] \tau_{h,j}u_l\, d\mathbf{x} \right| \\
&\leq c(c_2, p, n) \sum_j \int_\Omega \mu_{h,j} |\tau_{h,j}\mathbf{Du}| \eta |\nabla\eta| |\tau_{h,j}\mathbf{u}|\, d\mathbf{x} \\
&\quad + c(c_2, n) \sum_j \int_{\Omega_h} \mu_{h,j} |\mathbf{Du}|_{\mathbf{x}+h\mathbf{e}_j} |\tau_{h,j}\nabla(\eta^2)| |\tau_{h,j}u|\, d\mathbf{x} \\
&\leq c(c_1, c_2, p, n, \nabla\eta, \nabla^2\eta) \sum_j \int_{\Omega_h} \mu_{h,j}(|\mathbf{Du}|^2 + |\mathbf{Du}|^2_{\mathbf{x}+h\mathbf{e}_j} + |\tau_{h,j}\mathbf{u}|^2)\, d\mathbf{x} \\
&\quad + \epsilon(c_1, p) \sum_j \int_\Omega \eta^2 \mu_{h,j} |\tau_{h,j}\mathbf{Du}|^2\, d\mathbf{x}. \qquad (4.41)
\end{aligned}$$

The third integral breaks up naturally into two parts, which we will call IIIa and IIIb. For IIIa we have applying Hölder's inequality

$$\begin{aligned}
|\mathrm{IIIa}| &= \left| \int_\Omega T_{kl}(\mathbf{Du})(\partial_k\partial_j\eta^2)(\Psi_{jl} - \Psi_{lj})\, d\mathbf{x} \right| \\
&\leq c(\nabla^2\eta) \int_{\mathrm{supp}(\eta)} |\mathbf{T}(\mathbf{Du})|^{p'} + |\Psi|^p\, d\mathbf{x}. \qquad (4.42)
\end{aligned}$$

From (4.10) we have the estimate $|\mathbf{T}(\mathbf{Du})|^{p'} \leq c(c_2,p,n)(1+|\mathbf{Du}|^p)$. Also, we can estimate $|\Psi|^p \leq c(n)\sum_j |\tau_{-h,j}\sigma_{h,j}\mathbf{u}|^p$ to arrive at our final bound for IIIa,

$$\begin{aligned}|\mathrm{IIIa}| &\leq c(\nabla^2\eta)\int_{\mathrm{supp}(\eta)} c(c_2,p,n)(1+|\mathbf{Du}|^p) + c(n)\sum_j |\tau_{-h,j}\sigma_{h,j}\mathbf{u}|^p\,d\mathbf{x}\\ &\leq c(\nabla^2\eta)\int_{\mathrm{supp}(\eta)} c(c_2,p,n)(1+|\mathbf{Du}|^p) + c(n,\mathrm{supp}(\eta),\Omega)\sum_j |\nabla\sigma_{h,j}\mathbf{u}|^p\,d\mathbf{x}\end{aligned} \tag{4.43}$$

where we have used the fact that there exists a constant $c(\mathrm{supp}(\eta),\Omega)$ independent of h such that for all g in $W_0^{1,p}(\Omega)$,

$$\int_{\mathrm{supp}(\eta)} |\tau_{h,j}g|^p\,d\mathbf{x} \leq c(\mathrm{supp}(\eta),\Omega)\int_{\mathrm{supp}(\eta)} |\partial_j g|^p\,d\mathbf{x}.$$

For second part of integral III we have

$$\mathrm{IIIb} = \sum_j \int_\Omega (\partial_j\eta^2 T_{kl}(\mathbf{Du}) - \partial_l\eta^2 T_{kj}(\mathbf{Du}))\tau_{-h,j}\partial_k\sigma_{h,j}u_l\,d\mathbf{x}$$

We integrate the difference quotient by parts to get $\mathrm{IIIb} = \mathrm{IIIbi} + \mathrm{IIIbii}$ where

$$|\mathrm{IIIbi}| = \left|\sum_j \int_\Omega \partial_j\eta^2\tau_{h,j}T_{kl}(\mathbf{Du})\partial_k\sigma_{h,j}u_l - \partial_l\eta^2\tau_{h,j}T_{kj}(\mathbf{Du})\partial_k\sigma_{h,j}u_l)\,d\mathbf{x}\right|$$

and

$$|\mathrm{IIIbii}| = \left|\int_\Omega T_{kl}(\mathbf{Du})|_{\mathbf{x}+h\mathbf{e}_j}(\tau_{h,j}\partial_j\eta^2)\partial_k\sigma_{h,j}u_l - T_{kj}(\mathbf{Du})|_{\mathbf{x}+h\mathbf{e}_l}(\tau_{h,j}\partial_l\eta^2)\partial_k\sigma_{h,j}u_l\right|.$$

Using estimate (4.28) in Lemma 4.3 together with Hölder's inequality we easily see

$$\left|\sum_j \int_\Omega \partial_j\eta^2(\tau_{h,j}T_{kl}(\mathbf{Du})\partial_k\sigma_{h,j}u_l\,d\mathbf{x}\right| \leq \sum_j c(c_2,p,n)\int_\Omega \mu_{h,j}\eta|\nabla\eta||\tau_{h,j}\mathbf{Du}||\nabla\sigma_{h,j}\mathbf{u}|\,d\mathbf{x}. \tag{4.44}$$

The second term in IIIbi requires a bit more care before we can apply Lemma 4.3 directly. However, we may rewrite it using the Kronecker delta before applying (4.28) to get

$$\left|\sum_j \int_\Omega \partial_l\eta^2(\tau_{h,j}T_{km}(\mathbf{Du})\delta_{mj}\partial_k\sigma_{h,j}u_l\,d\mathbf{x}\right| \leq \sum_j c(c_2,p,n)\int_\Omega \mu_{h,j}\eta|\nabla\eta||\tau_{h,j}\mathbf{Du}||\nabla\sigma_{h,j}\mathbf{u}|\,d\mathbf{x}. \tag{4.45}$$

Combining (4.44) and (4.45) together with Young's inequality we have the estimate for IIIbi

$$|\text{IIIbi}| \le c(c_1, c_2, p, n) \sum_j \int_\Omega \mu_{h,j} |\nabla \eta|^2 |\nabla \sigma_{h,j} \mathbf{u}|^2 + \epsilon(c_1, p) \sum_j \int_\Omega \eta^2 \mu_{h,j} |\tau_{h,j} \mathbf{D}\mathbf{u}|^2 \, d\mathbf{x}. \tag{4.46}$$

On the other hand, IIIbii is easy to estimate as it has no singular terms. We may apply Hölder's inequality together with (4.10) to get

$$|\text{IIIbii}| \le \sum_j c(c_2, p, n, \nabla^2 \eta) \int_{\Omega_h} 1 + |\mathbf{D}\mathbf{u}|^p_{x+he_j} + |\nabla \sigma_{h,j} \mathbf{u}|^p \, d\mathbf{x} \tag{4.47}$$

Our last term to estimate is IV, which can be written

$$|\text{IV}| = \left| \sum_j \int_\Omega f_i \eta^2 \tau_{-h,j} \tau_{h,j} u_i + \sum_j \int_\Omega f_i \partial_j \eta^2 (\Psi_{ij} - \Psi_{ji}) \right|.$$

For the latter term we obtain from Hölder's inequality

$$\left| \sum_j \int_\Omega f_i \partial_j \eta^2 (\Psi_{ij} - \Psi_{ji}) \, d\mathbf{x} \right| \le \sum_j c(n) \int_\Omega \eta^2 |f|^2 + |\nabla \eta|^2 |\tau_{h,j} \sigma_{h,j} \mathbf{u}|^2 \, d\mathbf{x} \tag{4.48}$$

For the former, the estimate is completely standard from the theory of difference quotients, such as done in Chapter 3, and we have

$$\left| \sum_j \int_\Omega f_i \eta^2 \tau_{-h,j} \tau_{h,j} u_i \, d\mathbf{x} \right| \le c(c_1, p, \text{supp}(\eta), \Omega) \int_\Omega \eta^2 |f|^2 \, d\mathbf{x} + \\ + \epsilon(c_1, p) \sum_j \int_\Omega \eta^2 |\tau_{h,j} \partial_j \mathbf{u}|^2 \, d\mathbf{x} \tag{4.49}$$

We would like to bound the last term in (4.49) in terms of difference quotients of $\mathbf{D}\mathbf{u}$ so as to incorporate it into (4.36). To do this, we use (4.21) in (4.49) just as in the *a priori* estimate to conclude

$$\left| \sum_j \int_\Omega f_i \eta^2 \tau_{-h,j} \tau_{h,j} u_i \, d\mathbf{x} \right| \le c(c_1, p, \text{supp}(\eta), \Omega) \int_\Omega \eta^2 |f|^2 \, d\mathbf{x} \\ + c(c_1, p) \sum_j \int_\Omega |\nabla \eta|^2 |\tau_{h,j} u|^2 \, d\mathbf{x} + \epsilon(c_1, p) \sum_j \int_\Omega \eta^2 |\tau_{h,j} \mathbf{D}\mathbf{u}|^2. \tag{4.50}$$

Now we may combine (4.40), (4.41), (4.43), (4.46), (4.47), (4.48), and (4.50) together with the lower bound (4.36) to arrive at the estimate for the difference quotients

$$\sum_j \int_\Omega \mu_{h,j} |\tau_{h,j} \mathbf{Du}|^2 \, d\mathbf{x} \leq c(c_1, p, n, \nabla\eta, \nabla^2\eta, \mathrm{supp}(\eta), \Omega) \Bigg[\int_\Omega \eta^2 |\mathbf{f}|^2 \, d\mathbf{x} + \sum_j \int_{\Omega_h} |\nabla \sigma_{h,j} u|^2 + |\nabla \sigma_{h,j} u|^p + \mu_{h,j} \left[|\mathbf{Du}|^2 + |\mathbf{Du}|^2_{\mathbf{x}+h\mathbf{e}_j} + |\tau_{h,j}\mathbf{u}|^2 + |\nabla \sigma_{h,j}\mathbf{u}|^2 \right] \, d\mathbf{x} \Bigg]. \tag{4.51}$$

Using the absolute continuity of the integral together with the strong L^p convergence of $\tau_{h,k}\mathbf{u}$ to $\partial_k \mathbf{u}$ and the L^p boundedness of $\sigma_{h,j}\partial_j \mathbf{u}$ proven in Lemma 4.1 we see that the right hand side of (4.51) is uniformly bounded from above as $h \to 0$. Since the left-hand side of (4.51) bounds $\sum_j \int_\Omega \eta^2 |\tau_{h,j} \mathbf{Du}|^2 \, d\mathbf{x}$ from above, and since η is an arbitrary interior cutoff function,we conclude that for each j, $\partial_j \mathbf{Du} \in L^2_{loc}(\Omega)$. Employing (4.21) we conclude further that $\mathbf{u} \in W^{2,2}_{loc}(\Omega)$. Actually, $\mathbf{u}$ is even more regular than this. Since $\eta^2 \mu_{h,j} |\tau_{h,j} Du|^2$ is bounded in $L^1(\Omega)$ we conclude that some subsequence of $\eta\sqrt{\mu_{h,j}}\tau_{h,j} D_{kl}\mathbf{u}$ converges weakly in $L^2(\Omega)$. From the strong convergence of $\sqrt{\mu_{h,j}}$ in L^2 to μ and the strong convergence in L^2 of $\tau_{h,j} D_{kl}\mathbf{u}$ to $\partial_j D_{kl}\mathbf{u}$ we see that the weak limit must be $\eta\sqrt{\mu}\partial_j D_{kl}\mathbf{u}$ and therefore

$$\eta\sqrt{\mu}\partial_j D_{kl}\mathbf{u} \in L^2(\Omega). \tag{4.52}$$

However, the bounds given in (4.51) depend on the second derivatives of η among other things and are therefore not as sharp as the the desired estimate (4.34). We now use (4.52) to obtain (4.34).

Part 2. From (4.52) together with the growth estimate (4.3) it follows that $\partial_j T_{ij}(\mathbf{Du})$ is in $L^{p'}_{loc}(\Omega)$. From this, we are able to integrate the weak formulation (4.8) by parts to conclude that

$$-\int_\Omega \partial_j T_{ij}(\mathbf{Du})\phi_i \, d\mathbf{x} = \int_\Omega f_i \phi_i \, d\mathbf{x}. \tag{4.53}$$

for all ϕ in $\mathcal{J}$ and by continuity for all ϕ in $J^p(\Omega')$ every open Ω' with closure in Ω. Consider the test function $\phi \in W_0^{1,2}(\Omega)$ defined by

$$\begin{aligned}\phi_i &= \sum_j \left[-\eta^2 \partial_j \tau_{h,j} u_i + \partial_j \eta^2 \tau_{h,i} u_j - \partial_j \eta^2 \tau_{h,j} u_i\right] \\ &= -\sum_j \left[\partial_j \left(\eta^2 \tau_{h,j} u_i\right)\right] + \partial_j \eta^2 \tau_{h,i} u_j \end{aligned} \tag{4.54}$$

An easy computation shows that ϕ is also divergence free and lies in $L^p(\Omega)$ and therefore is a legitimate test-function for (4.53). Indeed a simple mollification argument shows that it is in the closure in $L^p(\Omega)$ of $\mathcal{J}$. Therefore, introducing it in (4.53), we see

$$\sum_j \int_\Omega \partial_k T_{ki}(\mathbf{Du}) \partial_j \left(\eta^2 \tau_{h,j} u_i\right) \, d\mathbf{x} = \sum_j \int_\Omega \partial_i T_{ij}(\mathbf{Du}) \partial_k \eta^2 \tau_{h,j} u^k \, d\mathbf{x} + \int_\Omega f_i \phi_i \, d\mathbf{x} \tag{4.55}$$

We would like to integrate both derivatives by parts in the left hand side of (4.55). This is justified since $T_{ij}(\mathbf{Du})$ is in $W^{1,p'}_{loc}$ and $\eta^2 \tau_{h,j} u_i$ is in $W_0^{1,p}$, and the result is

$$\begin{aligned}\sum_j \int_\Omega \eta^2 \partial_j T_{ki}(\mathbf{Du}) \partial_k \tau_{h,j} u_i \, d\mathbf{x} &= \sum_j \left[\int_\Omega \partial_i T_{ij}(\mathbf{Du}) \partial_k \eta^2 \tau_{h,j} u^k - \partial_j T_{ki}(\mathbf{Du}) \tau_{h,j} u_i \partial_k \eta^2 \, d\mathbf{x}\right] \\ &\quad + \int_\Omega f_i \phi_i \, d\mathbf{x}\end{aligned} \tag{4.56}$$

We now take the limit as h tends to zero. None of the terms on the right hand side present any difficulty but we must be careful with the term on the left. Since $\eta \partial_j T_{ki}(\mathbf{Du})$ is in $L^{p'}$ and since we have established $\eta \partial_k \tau_{h,j} u_i$ converges strongly only in L^2 we cannot immediately use continuity arguments. However, we may write each integrand as

$$\frac{\eta}{\sqrt{\mu}} \partial_j T_{ki}(\mathbf{Du}) \eta \sqrt{\mu} \partial_k \tau_{h,j} u_i.$$

From (4.3) we see that

$$
\begin{aligned}
\frac{\eta}{\sqrt{\mu}}|\partial_j T_{ki}(\mathbf{Du})| &= \frac{\eta}{\sqrt{\mu}}|\partial_{rs} T_{ki}(\mathbf{Du})\partial_j D_{rs}\mathbf{u}| \\
&\leq c(c_2)\eta\frac{1+|\mathbf{Du}|^{p-2}}{\sqrt{\mu}}|\partial_j \mathbf{Du}| \\
&\leq c(c_2)\eta\sqrt{\mu}|\partial_j \mathbf{Du}|
\end{aligned} \tag{4.57}
$$

which implies $\frac{\eta}{\sqrt{\mu}}\partial_j T_{ki}(\mathbf{Du})$ is in L^2. From the weak convergence in L^2 of $\eta\sqrt{\mu_{h,j}}\partial_k \tau_{h,j} u_i$ to $\eta\sqrt{\mu}\partial_k\partial_j u_i$ and the strong convergence of $\sqrt{\mu_{h,j}}$ to $\sqrt{\mu}$ in L^2 we see $\eta\sqrt{\mu}\partial_k \tau_{h,j} u_i$ converges weakly in L^2 to $\eta\sqrt{\mu}\partial_k\partial_j u_i$. Therefore, we are justified in taking this limit on the left-hand side of (4.56) and we arrive at

$$
\begin{aligned}
\int_\Omega \eta^2 \partial_j T_{ki}(\mathbf{Du})\partial_k\partial_j u_i \, d\mathbf{x} = &\int_\Omega \partial_i T_{ij}(\mathbf{Du})\partial_k \eta^2 \partial_j u^k - \partial_j T_{ki}(\mathbf{Du})\partial_j u_i \partial_k \eta^2 \, d\mathbf{x} \\
&+ \int_\Omega f_i \left(-\eta^2 \partial_j\partial_j u_i + \partial_j \eta^2 \partial_i u_j - \partial_j \eta^2 \partial_j u_i\right).
\end{aligned} \tag{4.58}
$$

Now we estimate from above and below similar to when we when we proved the existence of second derivatives. However, the estimates are cleaner and easier. The lower bound follows directly from (4.2) and we have (using the fact that $\mathbf{T}$ is symmetric)

$$
\begin{aligned}
\int_\Omega \eta^2 \partial_j T_{ki}(\mathbf{Du})\partial_k\partial_j u_i \, d\mathbf{x} &= \int_\Omega \eta^2 \partial_j T_{ki}(\mathbf{Du})\partial_j D_{ki}\mathbf{u} \, d\mathbf{x} \\
&\geq c_1 \sum_j \int_\Omega \eta^2 (1+|\mathbf{Du}|^{p-2})|\partial_j \mathbf{Du}|^2 \, d\mathbf{x}.
\end{aligned} \tag{4.59}
$$

For the first integral on the right-hand side of (4.58) we use (4.3) together with Young's inequality to obtain

$$
\begin{aligned}
&\left|\int_\Omega \partial_i T_{ij}(\mathbf{Du})\partial_k \eta^2 \partial_j u^k - \partial_j T_{ki}(\mathbf{Du})\partial_j u_i \partial_k \eta^2 \, d\mathbf{x}\right| \leq \\
&\qquad c(c_1, c_2, n)\int_\Omega (1+|\mathbf{Du}|^{p-2})|\nabla\eta|^2|\mathbf{Du}|^2 \, d\mathbf{x} + \\
&\qquad\qquad + \epsilon(c_1)\sum_j \int_\Omega \eta^2(1+|\mathbf{Du}|^{p-2})|\partial_j \mathbf{Du}|^2 \, d\mathbf{x}.
\end{aligned} \tag{4.60}
$$

The second integral on the right-hand side of (4.58) requires an application of Hölder's inequality together with (4.21) to get

$$\left|\int_\Omega f_i\left(-\eta^2\partial_j\partial_j u_i + \partial_j\eta^2\partial_i u_j - \partial_j\eta^2\partial_j u_i\right)\,d\mathbf{x}\right| \leq$$
$$c(c_1)\int_\Omega \eta^2|\mathbf{f}|^2\,d\mathbf{x} + c(c_1,n)\int_\Omega |\nabla\eta|^2|\nabla\mathbf{u}|^2\,d\mathbf{x}+$$
$$+\,\epsilon(c_1)\sum_j\int_\Omega \eta^2(1+|\mathbf{Du}|^{p-2})|\partial_j\mathbf{Du}|^2\,d\mathbf{x}. \quad (4.61)$$

Combining (4.59), (4.60) and (4.61) together with a final application of (4.21) gives us the estimate

$$\int_\Omega \eta^2\left[|\nabla^2\mathbf{u}|^2 + \sum_j(1+|\mathbf{Du}|^{p-2})|\partial_j\mathbf{Du}|^2\right]\,d\mathbf{x}$$
$$\leq c(c_1)\int_\Omega \eta^2|\mathbf{f}|^2\,d\mathbf{x} + c(c_1,c_2,n)\int_\Omega |\nabla\eta|^2|\left[|\nabla\mathbf{u}|^2 + (1+|\mathbf{Du}|^{p-2})\mathbf{Du}|^2\right]\,d\mathbf{x}$$

from which (4.34) follows easily.

□

Chapter 5
Applications to non-Newtonian Fluids

In this Chapter we apply Theorem 4.1 to the steady and unsteady motion of an incompressible fluid with equation of motion (1.1). We return our attention to the specific model (1.1) in which the stress tensor $\mathbf{T}$ can be written

$$T_{ij} = 2(\nu_1 + \nu_2 |\mathbf{Du}|^2) D_{ij}\mathbf{u}.$$

5.1 The Steady Case

Our first object of study is the steady case

$$\begin{aligned} -\partial_j(2(\nu_1 + \nu_2|\mathbf{Du}|^2)D_{ij}\mathbf{u}) &= -u_j\partial_j u_i - \partial_i \pi + f_i \\ \partial_i u_i &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{5.1}$$

Since the weak solutions of the Stokes-like system (2.1) come from such a regular space, $W_0^{1,4}(\Omega)$, the inertial term poses little problem for the regularity theory. Indeed, we define a weak solution of (5.1) to be a function $\mathbf{u}$ in $J_0^{1,4}(\Omega)$ such that

$$\int_\Omega 2(\nu_1 + \nu_2|\mathbf{Du}|^2)D_{ij}\mathbf{u}\partial_i\phi_j \, d\mathbf{x} = \int_\Omega -u_j\partial_j u_i\phi_i + f_i\phi_i \, d\mathbf{x} \tag{5.2}$$

for all ϕ in $\mathcal{J}$. Its regularity follows directly from Theorem 4.1.

Theorem 5.1 *Let Ω be a bounded open subset of $\mathbf{R}^3$. If $\mathbf{f}$ is in $L^2(\Omega)$ then every weak solution* *of (5.2) is in* $W^{2,2}_{loc}(\Omega)$ *and there exists a pressure* π *in* $W^{1,\frac{4}{3}}_{loc}(\Omega)$ *such that* $\mathbf{u}$ *and* π *satisfy (5.1) almost everywhere in* Ω.

Proof:

If a weak solution $\mathbf{u}$ exists, then from Sobolev embedding of $W^{1,4}_0(\Omega)$ into $C(\Omega)$ we see that $\mathbf{u}$ is in $L^\infty(\Omega)$ an therefore the term $u_j \partial_j u_i$ is in $L^4(\Omega)$ and hence also $L^2(\Omega)$. Therefore $\mathbf{u}$ is a weak solution of system (2.1) with right hand side

$$F_i = -u_j \partial_j u_i + f_i$$

in $L^2(\Omega)$. From 4.1 we see then that $\mathbf{u} \in W^{2,2}_{loc}(\Omega)$. From Section 2.2 we recall that there exists a pressure π in $L^{\frac{4}{3}}(\Omega)$ such that for all ϕ in $C^\infty_0(\Omega)$,

$$\int_\Omega 2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u} \partial_j \phi_i + \pi \ (\nabla \cdot \phi) - F_i \phi_i \, d\mathbf{x} = 0$$

Integrating this equation by parts we obtain

$$\int_\Omega \left[\partial_j (2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u}) + F_i\right] \phi_i \, d\mathbf{x} = \int_\Omega \pi \ (\nabla \cdot \phi) \, d\mathbf{x}. \tag{5.3}$$

Therefore, π is in $W^{1,\frac{4}{3}}_{loc}(\Omega)$ and

$$-\partial_j (2(\nu_1 + \nu_2 \left|\mathbf{Du}\right|^2) D_{ij}\mathbf{u}) = -\partial_i \pi + -u_j \partial_j u_i + f_i$$

almost everywhere.

□

Therefore, to complete the theory we need only show that weak solutions exist. We do this along the same lines as can be done for stationary solutions of the Navier-Stokes equations using the Leray-Schauder fixed point theorem. Our proof is different from that used at the end of Chapter 3 for solutions of the stationary Vector Burgers-like equation since we do not have a boundary proof and therefore cannot use the compact embedding of $W^{2,2}(\Omega)$ into $W^{1,4}(\Omega)$.

Theorem 5.2 *There exist weak solutions of (5.1).*

Proof:

We show solutions of (5.2) exist from the unique solvability of (2.1) and the Leray-Schauder fixed point theorem. Indeed, consider the function $\mathcal{F}_t$ defined by the map that takes $\mathbf{v}$ in $W_0^{1,p}(\Omega)$ to the unique $\mathbf{u}$ in $W_0^{1,p}(\Omega)$ that solves

$$\int_\Omega 2(\nu_1 + \nu_2|\mathbf{Du}|^2)D_{ij}\mathbf{u}D_{ij}\phi\, d\mathbf{x} = \int_\Omega (-v_j\partial_j v_i + f_i)\,\phi_i\, d\mathbf{x}.$$

for all ϕ in $W_0^{1,p}(\Omega)$. We show now that this map is compact, so let us suppose that $\{\mathbf{v}^r\}$ is a sequence in $W_0^{1,p}(\Omega)$ that converges weakly in $W_0^{1,p}(\Omega)$ and therefore strongly in $L^4(\Omega)$. Let $\mathbf{u}^r = \mathcal{F}(\mathbf{v}^r)$. Then it follows from (5.2) and an integration by parts that

$$\int_\Omega \big(2(\nu_1 + \nu_2|\mathbf{Du}^r|^2)D_{ij}\mathbf{u}^r D_{ij}(\mathbf{u}^r - \mathbf{u}^s) - 2(\nu_1 + \nu_2|\mathbf{Du}^s|^2)D_{ij}\mathbf{u}^s D_{ij}(\mathbf{u}^r - \mathbf{u}^s)\big)\ d\mathbf{x} =$$
$$\int_\Omega -(v_j^r - v_j^s)\partial_j v_i^r(v_i^r - v_i^s) - v_j^s(\partial_j(v_i^r - v_i^s))(v_i^r - v_i^s)\, d\mathbf{x} \quad (5.4)$$

From (5.4) together with Corollary 4.1 (or Appendix A), Korn's inequality [Neč66], and Hölder's inequality we see

$$c(\Omega)(\nu_1||\nabla(\mathbf{u}^r - \mathbf{u}^s)||_2^2 + \nu_2||\nabla(\mathbf{u}^r - \mathbf{u}^s)||_4^4) \leq$$
$$||\nabla\mathbf{v}^r||||\mathbf{v}^r - \mathbf{v}^s||_4^2 + c(\Omega,\nu_1)||\mathbf{v}^s||_\infty^2||\mathbf{v}^r - \mathbf{v}^s||^2$$

which, recalling Sobolev embedding, shows from the uniform boundedness of $||\mathbf{v}^r||_2$ and $||\mathbf{v}^r||_\infty$ and the strong convergence of $\{\mathbf{v}^r\}$ in $L^4(\Omega)$ that $\{\mathbf{u}^r\}$ is Cauchy in $W_0^{1,4}$ and therefore also the desired compactness. To apply the Leray-Schauder principle we need only show now that any solution of $\mathbf{u} = \lambda\mathcal{F}(\mathbf{u})$ is bounded uniformly for $\lambda \in [0,1]$. However, since $\int_\Omega u_j\partial_j u_i u_i$ vanishes we obtain from (4.9) and Korn's inequality the estimate

$$||\nabla\mathbf{w}||_4^3 \leq c(\Omega)\lambda^3||\mathbf{f}||_{(W_0^{1,p})^*}$$

which implies the existence of solutions of (5.2).

□

5.2 The Unsteady Case

We now turn our attention to showing the existence of regular solutions of (1.1).

Theorem 5.3 *Let Ω be a bounded domain in $\mathbf{R}^3$, let $\mathbf{f}$ be in $L^2(0,T;L^2(\Omega))$, let $\mathbf{u}_0$ be in J. Then there exist functions $\mathbf{u}$ and π that satisfy (1.1) almost everywhere in $\Omega \times [0,T]$ with*

$$||\mathbf{u}(t) - \mathbf{u}_0|| \to 0$$

as $t \to 0^+$. Moreover, $\mathbf{u} \in L^4(0,T;J_0^{1,4})$ and $\sqrt{t}\mathbf{u} \in W^{1,2}(\Omega \times [0,T]) \cap L^2(0,T;W^{2,2}(\Omega'))$ for every open set Ω' with closure contained in Ω.

Let $\{\mathbf{a}_k\}$ be a basis for $J_0^{1,4}(\Omega)$ that we will take to be orthogonal in $J(\Omega)$. To begin, we construct Galerkin solutions in the usual way. We let $\mathbf{u}^k = \sum_{j=1}^k c^{j,k}\mathbf{a}^j$ where the functions $c^{j,k}$ satisfy the ordinary differential equation

$$\frac{d}{dt}c^{j,k} = -\sum_{l,m=1}^{k} c^{l,k}c^{m,k}(\mathbf{a}^l \cdot \nabla \mathbf{a}^m, \mathbf{a}^j) - (D_{lm}\mathbf{u}^k, D_{lm}\mathbf{a}^j) + (\mathbf{f}, \mathbf{a}_j) \tag{5.5}$$

with initial condition $c^{j,k}(0) = (\mathbf{a}^j, \mathbf{u}_0)$. By the standard theory of ordinary differential equations, we find that the $c^{j,k}$ exist on some interval $[0, T_k]$.

Multiplying (5.5) by $c^{j,k}$ and summing on j we obtain in the same way as for the Navier-Stokes equations the energy estimate for the Galerkin approximations

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}^k||^2 + (2\left(\nu_1 + \nu_2|\mathbf{D}\mathbf{u}^k|^2\right) D_{ij}\mathbf{u}^k, D_{ij}\mathbf{u}^k) = (\mathbf{f}, \mathbf{u}^k). \tag{5.6}$$

From Gronwall's inequality and (4.9) we arrive in the usual way at

$$||\mathbf{u}^k(t)||^2 + \int_0^t \int_\Omega 2\nu_1|\mathbf{D}\mathbf{u}^k|^2 + 2\nu_2|\mathbf{D}\mathbf{u}|^4 \, d\mathbf{x}\, ds \le B_1 \tag{5.7}$$

where B_1 is finite and depends only on $||\mathbf{u}_0||$ and $||\mathbf{f}||_{L^2(0,T_k;L^2(\Omega))}$. Therefore, we can extend the interval of existence of the $c^{j,k}$ to any interval $[0,T]$. We shall now work with some arbitrary but fixed $T > 0$.

The second estimate, roughly based on an *a priori* estimate found in [Hey93], follows from multiplying (5.5) by $\frac{d}{dt}c^{j,k}$ and summing on j to get, using $\dot{\mathbf{u}}$ to denote $\frac{d}{dt}\mathbf{u}$,

$$(2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}^k|^2\right)D_{lm}\mathbf{u}^k, D_{lm}\dot{\mathbf{u}}^k)+\frac{3}{4}||\dot{\mathbf{u}}^k||^2 \leq ||\mathbf{u}^k\cdot\nabla\mathbf{u}^k||_2^2+||\mathbf{f}||_2^2. \tag{5.8}$$

We would now like to bound the inertial term using the first estimate (5.7) and apply Gronwall's inequality to (5.8). First we show that $(2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}^k|^2\right)D_{lm}\mathbf{u}^k, D_{lm}\dot{\mathbf{u}}^k)$ is a derivative. Indeed, $2(D_{lm}\mathbf{u}^k, D_{lm}\dot{\mathbf{u}}^k) = \frac{d}{dt}||\mathbf{D}\mathbf{u}^k||^2$ and $2(|\mathbf{D}\mathbf{u}|^2 D_{lm}\mathbf{u}^k, D_{lm}\dot{\mathbf{u}}^k) = \frac{1}{4}\frac{d}{dt}||\mathbf{D}\mathbf{u}||_4^4$ and therefore

$$\begin{aligned}(2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}^k|^2\right)D_{lm}\mathbf{u}^k, D_{lm}\dot{\mathbf{u}}^k) &= \frac{d}{dt}\left[\nu_1||\mathbf{D}\mathbf{u}^k||^2+\frac{\nu_2}{2}||\mathbf{D}\mathbf{u}^k||_4^4\right]\\ &= \frac{d}{dt}\mathcal{F}(\mathbf{D}\mathbf{u}^k).\end{aligned} \tag{5.9}$$

Multiplying (5.8) by t we obtain using (5.9),

$$\frac{d}{dt}\left[t\mathcal{F}(\mathbf{D}\mathbf{u}^k)\right]+\frac{3t}{4}||\dot{\mathbf{u}}^k||^2 \leq t||\mathbf{u}^k\cdot\nabla\mathbf{u}^k||_2^2+t||\mathbf{f}||_2^2+\mathcal{F}(\mathbf{D}\mathbf{u}^k(0)). \tag{5.10}$$

We now want to bound the convective term in such a way that we can apply Gronwall's inequality. As in the stationary case, we can use $W^{1,4}$ control to our advantage here. From Sobolev's inequality we have $||\mathbf{u}^k\cdot\nabla\mathbf{u}^k|| \leq ||\nabla\mathbf{u}^k||||\nabla\mathbf{u}^k||_3$. From the boundedness of Ω and Korn's inequality it follows that

$$||\mathbf{u}^k\cdot\nabla\mathbf{u}^k|| \leq c(\Omega)||\mathbf{D}\mathbf{u}||_4^4. \tag{5.11}$$

Thus, from Gronwall's inequality applied to (5.10) combined with (5.11) we see

$$t\nu_1||\mathbf{D}\mathbf{u}^k||_2^2+t\nu_2||\mathbf{D}\mathbf{u}^k||_4^4+\int_0^t s||\dot{\mathbf{u}}^k(s)||_2^2\,ds. \leq B_2(\Omega, B_1) \tag{5.12}$$

We now turn to the convergence properties of the sequence $\mathbf{u}^k$. As is usual, we shall speak of convergence of subsequences of $\mathbf{u}^k$ as convergence of the sequence itself. Since our estimates do not give us control in advance of the second derivatives, we are not able to to prove convergence properties as simply as in the case of the spectral-Galerkin method for the Navier-Stokes equations. On the other hand, since we have more information than is traditional for weak solutions, we are able to prove the estimates more easily than those for weak solutions. From (5.7) we have

$$\int_{\Omega_T} |\mathbf{u}^k|^2 \, d\mathbf{x} \, dt \leq T B_1,$$

so $\mathbf{u}^k$ converges weakly in $L^2(\Omega_T)$ to some $\mathbf{u}$. From bounds (5.7) and (5.12) we have

$$\int_{\Omega \times [\delta, T]} |\mathbf{u}^k|^2 + |\dot{\mathbf{u}}^k|^2 + |\nabla \mathbf{u}^k|^2 \leq (1+T) B_1 + \frac{1}{\delta} B_2$$

and so $\mathbf{u}^k$ converges weakly in $W^{1,2}(\Omega_{[\delta,T]})$ and therefore strongly in $L^2(\Omega_{[\delta,T]})$ to $\mathbf{u}$. Furthermore,

$$\begin{aligned} \int_{\Omega_T} |\mathbf{u} - \mathbf{u}^k|^2 &= \int_{\Omega \times [0,\delta]} |\mathbf{u} - \mathbf{u}^k|^2 + \int_{\Omega \times [\delta,T]} |\mathbf{u} - \mathbf{u}^k|^2 \\ &\leq 2||\mathbf{u}||^2_{L^2(\Omega \times [0,\delta])} + 2||\mathbf{u}^k||^2_{L^2(\Omega \times [0,T])} + \int_{\Omega \times [\delta,T]} |\mathbf{u} - \mathbf{u}^k|^2 \\ &\leq 2||\mathbf{u}||_{L^2(\Omega \times [0,\delta])} + 2\delta B_1 + \int_{\Omega \times [\delta,T]} |\mathbf{u} - \mathbf{u}^k|^2. \end{aligned} \tag{5.13}$$

Taking limits in k we have using the strong convergence of $\mathbf{u}^k$ on $\Omega \times [\delta, T]$ that $\limsup_{k \to \infty} ||\mathbf{u}^k - \mathbf{u}||^2_{L^2(\Omega_T)} \leq c2||\mathbf{u}||_{L^2(\Omega \times [0,\delta])} + \delta B_1$. Since δ is arbitrary we have shown that $\{\mathbf{u}^k\}$ converges strongly in $L^2(\Omega_T)$ to $\mathbf{u}$.

From the bound (5.12) we see that for every $t > 0$, $\{\mathbf{u}^k(t)\}$ converges weakly in $W_0^{1,4}(\Omega)$ to some $\mathbf{v}(t)$ and thus strongly in $L^4(\Omega)$ and $L^2(\Omega)$.

Let us show that $\mathbf{v}(s) = \mathbf{u}|_{t=s}$. In particular, since $\mathbf{u}$ is in $W^{1,2}(\Omega \times [\delta, T])$, it has trace values $\mathbf{u}(t)$ in $L^2(\Omega)$ for each $t > 0$. We need to show that $\mathbf{u}(t) = \mathbf{v}(t)$. However, if $\mathbf{w}$ is any function in $W_0^{1,2}(\Omega)$, then for $t_2 > t_1 > 0$ we have

$$\begin{aligned} &\int_\Omega (t_2 - t_1)(u_i(\mathbf{x}, t_2) - u_i^k(\mathbf{x}, t_2)) w_i(\mathbf{x}) \, d\mathbf{x} = \\ &\qquad \int_{\Omega \times [t_1, t_2]} (t - t_1)(\dot{u}_i - \dot{u}_i^k) w_i(x) + (u_i - u_i^k) w_i(x) \, d\mathbf{x} \, dt. \end{aligned} \tag{5.14}$$

Thus, by weak convergence of $\dot{\mathbf{u}}^k$ to $\dot{\mathbf{u}}$ in $L^2(\Omega\times[t_1,t_2])$ and by weak convergence of $\mathbf{u}^k$ to $\mathbf{u}$ in $L^2(\Omega\times[t_1,t_2])$ we have $\mathbf{u}^k(t_2)$ converges weakly in $L^2(\Omega)$ to $\mathbf{u}(t_2)$. Since $\mathbf{u}^k(t_2)$ converges strongly in $L^2(\Omega)$ to $\mathbf{v}(t_2)$, it must be that $\mathbf{v}(t_2)$ and $\mathbf{u}(t_2)$ are the same. Since t_2 is arbitrary, we have $\mathbf{v}(t)$ is in fact the trace value of $\mathbf{u}$ at t.

Now we turn to showing that $\mathbf{u}$ is a solution of the PDE. First we deal with the convective term. If ϕ is a smooth function,

$$\int_0^T\int_\Omega \mathbf{u}^k\cdot\nabla\mathbf{u}^k\phi-\mathbf{u}\cdot\nabla\mathbf{u}\phi\,d\mathbf{x}\,dt\le\int_0^T\int_\Omega|\mathbf{u}^k-\mathbf{u}||\nabla\mathbf{u}^k||\phi|+|\mathbf{u}||\nabla(\mathbf{u}^k-\mathbf{u})||\phi|\,d\mathbf{x}\,dt \tag{5.15}$$

By strong convergence of $\{\mathbf{u}^k\}$ to $\mathbf{u}$ in $L^2(\Omega_T)$ and by weak convergence of $\{\nabla\mathbf{u}^k\}$ to $\nabla\mathbf{u}$ in $L^2(\Omega_T)$ we see that $\mathbf{u}^k\cdot\nabla\mathbf{u}^k$ converges weakly in $L^2(\Omega_T)$ to $\mathbf{u}\cdot\nabla\mathbf{u}$.

Next we deal with the viscosity term. Since $\{\mathbf{Du}^k\}$ remains bounded in $L^4(\Omega_T)$ we have $2\left(\nu_1+\nu_2|\mathbf{Du}^k|^2\right)\mathbf{Du}$ remains bounded in $L^{\frac{4}{3}}(\Omega_T)$, and so converges weakly to some tensor $\mathcal{T}$ in $L^{\frac{4}{3}}(\Omega_T)$. We now show $\mathcal{T}=2\left(\nu_1+\nu_2|\mathbf{Du}|^2\right)\mathbf{Du}$ via a simple monotonicity technique. If ϕ^l is any function of the form $\sum_{j=1}^l b_j(t)\mathbf{a}^j$ with b_j continuous in t then (5.5) and (4.11) imply for $k\ge l$

$$\int_\delta^T\int_\Omega(\dot{\mathbf{u}}^k+\mathbf{u}^k\cdot\nabla\mathbf{u}^k-\mathbf{f})\cdot\phi^l\,d\mathbf{x}\,dt=-\int_\delta^T\int_\Omega 2\left(\nu_1+\nu_2|\mathbf{Du}^k|^2\right)D_{ij}\mathbf{u}^kD_{ij}\phi^l\,d\mathbf{x}\,dt.$$

Taking the limit in k using the weak convergence of $\dot{\mathbf{u}}^k$ to $\dot{\mathbf{u}}$, the weak convergence of $\mathbf{u}^k\cdot\nabla\mathbf{u}^k$ to $\mathbf{u}\cdot\nabla\mathbf{u}$, and the weak convergence of $\mathbf{T}(\mathbf{Du}^k)$ to $\mathcal{T}$ we see

$$\int_\delta^T\int_\Omega(\dot{\mathbf{u}}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})\cdot\phi^l\,d\mathbf{x}\,dt=-\int_\delta^T\int_\Omega\mathcal{T}_{ij}D_{ij}\phi^l\,d\mathbf{x}\,dt. \tag{5.16}$$

In particular, for $\phi^l=\mathbf{u}^l$ we have

$$\int_\delta^T\int_\Omega(\dot{\mathbf{u}}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})\cdot\mathbf{u}^l\,d\mathbf{x}\,dt=-\int_\delta^T\int_\Omega\mathcal{T}_{ij}D_{ij}\mathbf{u}^l\,d\mathbf{x}\,dt.$$

Taking the limit in l using the strong convergence of $\mathbf{u}^l$ to $\mathbf{u}$ in $L^2(\Omega_T)$ and the weak convergence of $\nabla\mathbf{u}^l$ to $\nabla\mathbf{u}$ in $L^4(\Omega_T)$ and the fact that $\int_\Omega\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\,d\mathbf{x}$ vanishes we see

$$\int_\delta^T\int_\Omega(\dot{\mathbf{u}}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})\cdot\mathbf{u}^l\,d\mathbf{x}\,dt=-\int_\delta^T\int_\Omega\mathcal{T}_{ij}D_{ij}\mathbf{u}^l\,d\mathbf{x}\,dt. \tag{5.17}$$

From (4.9) we have

$$\int_\delta^T \int_\Omega \left[2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}^k|^2\right) D_{ij}\mathbf{u}^k - 2\left(\nu_1+\nu_2|\mathbf{D}\phi^l|^2\right) D_{ij}\phi^l\right] D_{ij}(\mathbf{u}^k-\phi^l)\,d\mathbf{x}\,dt \geq 0$$

and therefore, from (5.5),

$$\int_\delta^T \int_\Omega -2\left(\nu_1+\nu_2|\mathbf{D}\phi^l|^2\right) D_{ij}(\mathbf{u}^k-\phi^l) + (\dot{\mathbf{u}}^k+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{f})\phi^l - (\dot{\mathbf{u}}^k - f)\mathbf{u}^k\,d\mathbf{x}\,dt \geq 0.$$

Taking the limit as before (now also using the weak convergence of $\dot{\mathbf{u}}^k$ and the strong convergence of $\mathbf{u}$ in $L^2(\Omega\times[\delta,T])$ to handle the $\dot{\mathbf{u}}^k\mathbf{u}$ term) and using (5.17) we see

$$\int_\delta^T \int_\Omega (\mathcal{T}_{ij} - 2\left(\nu_1+\nu_2|\mathbf{D}\phi^l|^2\right)) D_{ij}(\mathbf{u}-\phi^l)\,d\mathbf{x}\,dt \geq 0$$

and by density of functions of the type ϕ^l in $L^4(\delta,T;J_0^{1,4}(\Omega))$ and continuity of the previous integral on ϕ^l in this space we see

$$\int_\delta^T \int_\Omega (\mathcal{T}_{ij} - 2\left(\nu_1+\nu_2|\mathbf{D}\phi^l|^2\right)) D_{ij}(\mathbf{u}-\phi)\,d\mathbf{x}\,dt \geq 0 \tag{5.18}$$

for all ϕ in $L^4(\delta,T;J_0^{1,4}(\Omega))$. Letting $\phi=\mathbf{u}-\epsilon\mathbf{a}^j$ and taking the limit in epsilon we see

$$\int_\delta^T \int_\Omega (\mathcal{T}_{ij} - 2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}|^2\right) D_{ij}\mathbf{u}) D_{ij}(\mathbf{u}-\phi)\,d\mathbf{x}\,dt = 0$$

Since the above is true also for $\phi=\mathbf{u}+\epsilon\mathbf{a}^j$ we determine that, in fact,

$$\int_\delta^T \int_\Omega (\mathcal{T}_{ij} - 2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}|^2\right) D_{ij}\mathbf{u}) D_{ij}(\mathbf{a}^j) = 0. \tag{5.19}$$

Let us now show that $\mathbf{u}$ satisfies the initial condition. From (5.5) we see

$$(\mathbf{u}^k(t),\mathbf{a}^l(t)) = (\mathbf{u}_0,\mathbf{a}^l(0)) + \int_0^t \int_\Omega -\mathbf{u}^k\cdot\nabla\mathbf{u}^k\cdot\mathbf{a}^l - 2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}^k|^2\right) D_{ij}\mathbf{u}^k D_{ij}\mathbf{a}^l + \mathbf{f}\cdot\mathbf{a}^l\,d\mathbf{x}\,dt$$

Taking the limit in k, using now also the strong convergence in $L^2(\Omega)$ of $\mathbf{u}^k$ to $\mathbf{u}$ we obtain

$$(\mathbf{u}(t),\mathbf{a}^l(t)) = (\mathbf{u}_0,\mathbf{a}^l(0)) + \int_0^t \int_\Omega -\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{a}^l - 2\left(\nu_1+\nu_2|\mathbf{D}\mathbf{u}|^2\right) D_{ij}\mathbf{u} D_{ij}\mathbf{a}^l + \mathbf{f}\cdot\mathbf{a}^l\,d\mathbf{x}\,dt.$$

Therefore $\mathbf{u}(t)$ converges to $\mathbf{u}(0)$ weakly in $L^2(\Omega)$. However, from the energy inequality (5.6) we see that $||\mathbf{u}^k(t)||^2$ tends to $||\mathbf{u}^k(0)||^2$ uniformly in t and therefore $||\mathbf{u}(t)||^2$ tends to $||\mathbf{u}_0||^2$. Thus we have established that $\mathbf{u}(t)$ converges strongly to $\mathbf{u}_0$ as $t \to 0^+$.

We are now in a position to show that $\mathbf{u}$ has second spatial derivatives. Although we were not able to control these in taking the limit of the $\mathbf{u}^k$, we have ample control over all other aspects of the equation to make this possible. In particular, if ξ is any continuous function, then letting $\phi^l = \xi(t)\mathbf{a}^l(x)$ in (5.16) we see, using (5.19),

$$\int_\delta^T \xi(t) \int_\Omega 2\left(\nu_1 + \nu_2|\mathbf{Du}|^2\right) D_{ij}\mathbf{u}D_{ij}\mathbf{a}^j + (\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\cdot\mathbf{a}^l \, d\mathbf{x}\, dt = 0.$$

Now since

$$\int_\Omega 2\left(\nu_1 + \nu_2|\mathbf{Du}|^2\right) D_{ij}\mathbf{u}D_{ij}\mathbf{a}^j + (\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\cdot\mathbf{a}^j \, d\mathbf{x}$$

is in $L^{\frac{4}{3}}(\delta, T)$ and continuous functions are dense in $L^4(\delta, T)$, the Rietz-Ritz representation theorem implies

$$\int_\Omega T_{ij}(\mathbf{Du})D_{ij}\mathbf{a}^j = -\int_\Omega (\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\cdot\mathbf{a}^j \, d\mathbf{x}$$

at almost every t in $[\delta, T]$. Repeating this process for each of the countably many functions $\mathbf{a}^j$, using the fact that a countable collection of zero measure sets has zero measure, we see that (5.2) holds for every basis function $\mathbf{a}^j$ for almost every t. Since $\dot{\mathbf{u}} + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f}$ is in $L^2(\Omega)$ for almost every t we obtain immediately from Theorem 4.1 and the fact that δ is arbitrary that at almost every $t > 0$, $\mathbf{u}$ is in $W^{2,2}_{loc}(\Omega)$. Moreover, at almost every t, the estimate (4.34) from 4.1 holds. However, we are not quite in a position to claim that $\mathbf{u} \in L^2(\delta, T; W^{2,2}(\Omega'))$ since we have not determined that $||\mathbf{u}||_{W^{2,2}(\Omega')}$ is measurable. However, since

$$\begin{aligned} t||\tau_{h,m}\eta^2\nabla\mathbf{u}||^2 &\leq c(\Omega', \Omega)t||\partial_m\nabla\mathbf{u}||^2_{L^2(\Omega')} \\ &\leq c(\Omega', \Omega)\left[t||\mathbf{f}||_2^2 + t||\dot{\mathbf{u}}||_2^2 + t||\mathbf{u}\cdot\nabla\mathbf{u}||_2^2 + t||\nabla\mathbf{u}||_2^2 + t||\mathbf{Du}||_4^4\right] \end{aligned} \quad (5.20)$$

we get from the Dominated Convergence Theorem and the almost everywhere convergence of $t||\tau_{h,m}\eta^2\nabla\mathbf{u}||$ to $t||\partial_m\eta^2\nabla\mathbf{u}||$ that $\sqrt{t}\mathbf{u} \in L^2(0,T;W^{2,2}(\Omega'))$ for every open set Ω' with closure contained in Ω. Finally, we obtain in the usual way from (5.16) and the regularity of $\mathbf{u}$ the existence of a π with $\nabla\pi \in L^{\frac{4}{3}}_{loc}(\Omega)$ such that $\mathbf{u}$ and π satisfy (1.1) almost everywhere.

□

We complete our discussion of the time-dependent case by mentioning that our solution is unique.[1] Indeed, let $\mathbf{v}$ and $\mathbf{u}$ be two such solutions and let $\mathbf{w}$ be their difference. Then we see that $\mathbf{w}$ satisfies

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + 2\int_\Omega (\nu_1 + \nu_2|\mathbf{Du}|^2)Du - (\nu_1 + \nu_2|\mathbf{Dv}|^2)\mathbf{Dv}\,d\mathbf{x} = -\int_\Omega w_i D_{ij}\mathbf{u}w_j\,d\mathbf{x}.$$

in L^1. Applying estimate (4.11) we see then that at almost every time

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + 2\nu_1||\mathbf{Dw}||^2 \leq -\int_\Omega w_i D_{ij}uw_j\,d\mathbf{x}.$$

We now use Hölder's inequality and a Sobolev inequality to get at almost every time

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + 2\nu_1||\mathbf{Dw}||^2 \leq c(\Omega)||\nabla\mathbf{w}||^{\frac{3}{2}}||\mathbf{w}||^{\frac{1}{2}}||\mathbf{Du}||$$

$$\leq \nu_1||\nabla\mathbf{w}||^2 + c(\Omega,\nu_1)||\mathbf{Du}||_4^4||\mathbf{w}||^2.$$

Thus we find

$$\frac{1}{2}\frac{d}{dt}||\mathbf{w}||^2 + \nu_1||\mathbf{Dw}||^2 \leq c(\Omega,\nu_1)||\mathbf{Du}||_4^4||\mathbf{w}||^2$$

and using the fact that $\mathbf{w}$ converges strongly in L^2 to 0 as t tends to 0 we apply Gronwall's inequality and find that $||\mathbf{w}|| = 0$ at almost every time. We have proven

Theorem 5.4 *The solutions of (1.1) with the regularity of those found in Theorem 5.3 are unique.*

[1] Actually, it is not hard to show that our solutions are also weak solutions in the sense of Ladyzhenskaya and are therefore unique by her arguments.

Chapter 6
Epilogue

We have presented in this thesis an approach for studying interior regularity of elliptic and parabolic systems with solenoidal constraint. Unfortunately, we have left many questions unanswered. What we have shown is that if $\mathbf{f}$ is in $L^2(\Omega \times [0,T])$ and $\mathbf{u}_0$ is in J, then there exists a solution of

$$\begin{aligned} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla \pi + 2\,\mathrm{div}((\nu_1 + \nu_2 |\mathbf{Du}|^2)\mathbf{Du}) + \mathbf{f} \\ \mathrm{div}\,\mathbf{u} &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0 \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 \end{aligned} \tag{6.1}$$

such that for every open set Ω' with closure contained in Ω $\mathbf{u}$ has second derivatives in $L^2(\Omega' \times [0,T])$ and that $t\mathbf{u}$ has a time derivative in $L^2(\Omega \times [0,T])$. Thus we naturally ask whether we can extend these results up to smooth portions of the boundary, and whether higher regularity of $\mathbf{u}$ can be obtained in time and space. We hope with these final words to briefly outline some of the difficulties involved in extending our results to answer these questions.

6.1 Boundary Regularity

We note from the method of proof in Theorem 5.3 that a boundary regularity proof (e.g. $\mathbf{u}$ in $W^{2,2}(\Omega)$) of the Stokes-like system

$$\begin{aligned} -2\partial_j \left(\left(\nu_1 + \nu_2 \left| \mathbf{Du} \right|^2 \right) D_{ij} \mathbf{u} \right) &= \partial_i \pi + f_i \\ \partial_i u_i &= 0 \\ \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned} \tag{6.2}$$

would translate immediately to boundary regularity of (6.1). The central difficulty for proving boundary regularity of(6.2) is the same as that of interior regularity, namely that we need more regularity from the pressure than that which arises naturally for weak solutions, as seen in Chapter 4. We avoided this difficulty in the interior case by using a solenoidal test function. However, we do not have even for the Stokes system a solenoidal test function approach for boundary regularity. Since such an approach to boundary regularity has not yet been obtained for the well studied Stokes system, it seems that seems that without a truly remarkable result we must work with the pressure. Recall that the troublesome term to estimate is

$$\int_{\Omega} \pi\eta \left(\nabla\eta \cdot \partial_i \partial_i \mathbf{u} \right) \, d\mathbf{x}.$$

So it would be sufficient to show that π is in $L^2(\Omega)$. From Theorem 4.1 we deduce that the pressure is in $W^{1,4/3}_{loc}(\Omega)$ as well as in $L^{\frac{4}{3}}(\Omega)$. Such a result by itself would certainly be useless in proving something about π up to the boundary since one can imagine many functions that are smooth in a domain but have a singularity of arbitrary growth at a boundary point (e.g. $1/|\mathbf{x} - \mathbf{x}_0|^q$ for $\mathbf{x}_0$ a boundary point of Ω). However, we also obtain from Theorem 4.1 a growth estimate which could say something about π up to the boundary. Let us take for the sake of exposition Ω to be the cube Q in $\mathbf{R}^3$ defined by $\{\mathbf{x} : |x_1| < 1, |x_2| < 1, 0 < x_3 < 2\}$. Let ξ_δ be a cut-off function on $[0, 2]$ that is 1 on $[\delta, 1]$. It is easy to show that we can find such functions ξ_δ such that $|\nabla \xi_\delta| \leq \frac{c}{\delta}$ for some fixed constant c. Let $\xi(x_1,x_2)$ be an arbitrary cut-off function on the unit ball of $\mathbf{R}^2$. Then $\eta_\delta(\mathbf{x})$ defined by $\xi(x_1,x_2)\xi_\delta(x_3)$ is a cut-off function on Q and if we define $Q_\delta\{\mathbf{x}\in Q : x_3>\delta\}$ we obtain from the growth estimate of Theorem 4.1

$$\begin{aligned}\int_{Q_\delta}|\nabla\pi|^{\frac{4}{3}}\,d\mathbf{x} &\leq c(c_1,c_2,n)\left(\int_{Q_\delta}\nu(\mathbf{Du})^2+\eta_\delta^2|\mathbf{f}|^2\,d\mathbf{x}+\int_{Q_\delta}|\nabla\eta_\delta|^2\left[|\nabla\mathbf{u}|^2+|\mathbf{Du}|^4\right]\,d\mathbf{x}\right)\\ &\leq c(c_1,c_2,n)\left(\int_{Q_\delta}\nu(\mathbf{Du})^2+|\mathbf{f}|^2\,d\mathbf{x}+\frac{1}{\delta^2}\int_{Q_1\backslash Q_\delta}\left[|\nabla\mathbf{u}|^2+|\mathbf{Du}|^4\right]\,d\mathbf{x}\right).\end{aligned}$$

All terms on the right hand side are uniformly bounded in δ except

$$\frac{1}{\delta^2}\int_{Q_1\backslash Q_\delta}\left[|\nabla\mathbf{u}|^2+|\mathbf{Du}|^4\right]\,d\mathbf{x}.$$

Therefore,

$$\lim_{\delta\to 0}\delta^2\int_{Q_\delta}|\nabla\pi|^{\frac{4}{3}}\,d\mathbf{x}=0.$$

We are lead to ask, then, what sort of boundary singularity in π might arise in growth of this type. One might hope that this restricts π to be in a smaller space than $L^{\frac{4}{3}}(\Omega)$. However, consider the function $\pi_{singular}$ given by $\pi=\frac{1}{|\mathbf{x}|^{(\frac{11}{4}-\epsilon)}}$. Then a simple calculation shows

$$\int_{Q_\delta}|\nabla\pi_{singular}|^{\frac{4}{3}}\,d\mathbf{x}=o(\frac{1}{\delta^2}).$$

for every $\epsilon>0$.

Since $1/|\mathbf{x}|^{(\frac{11}{4}-\epsilon)}$ isn't even in $L^{\frac{4}{3}}$ for ϵ small, we see that our restriction does not, in this approach, offer useful information at the boundary.

We are not entirely without hope, however. We note that we do not even need that π lies in L^2. If $\pi/\sqrt{\nu(\mathbf{Du})}$ is in L^2, then we can estimate this term by

$$\int_\Omega\pi\eta\left(\nabla\eta\cdot\partial_i\partial_i\mathbf{u}\right)\,d\mathbf{x}\leq c(\epsilon)\int_\Omega|\nabla\eta|^2\frac{\pi^2}{\nu(\mathbf{Du})}\,d\mathbf{x}+\epsilon\int_\Omega\eta^2\nu(\mathbf{Du})|\partial_i\partial_i\mathbf{u}|^2\,d\mathbf{x},$$

which would be sufficient to incorporate the final term into the lower bound obtained by ellipticity. Unfortunately, an estimate of this type would be hard to obtain (even though it is weaker that an L^2 estimate) since it involves a very strong pairing between the growth of π and $\nabla\mathbf{u}$.

6.2 Higher Time Regularity

To get study higher time regularity one might expect to proceed as in the Navier-Stokes equations and take derivatives of (6.1) to obtain a sequence of inequalities to use in Gronwall's inequality and thereby control growth of higher time derivatives. For example, this has already been done for a single step higher in [Hey93]. To see how this works, we will assume for simplicity that $\mathbf{f}$ is 0. Then taking the derivative of (6.1) with respect to time, multiplying by $\dot{\mathbf{u}}$ and integrating over Ω we see

$$\frac{1}{2}||\dot{\mathbf{u}}||^2 + 2\int_\Omega (\nu_1 \mathbf{D}\dot{\mathbf{u}} + \nu_2 \mathbf{D}\mathbf{u} : \mathbf{D}\dot{\mathbf{u}}\mathbf{D}\mathbf{u})\,\mathbf{D}\dot{\mathbf{u}}\,d\mathbf{x} = -\int_\Omega u_{i,t} D_{ij}\mathbf{u} u_{j,t}\,d\mathbf{x}.$$

By a Sobolev inequality we see

$$\begin{aligned}\int_\Omega u_{i,t}\partial_i u_j u_{j,t}\,d\mathbf{x} &\leq c||\dot{\mathbf{u}}||_4^2||\mathbf{D}\mathbf{u}|| \\ &\leq \frac{\nu_1}{2}||\nabla\dot{\mathbf{u}}||^2 + c(\nu_1)||\mathbf{D}\mathbf{u}||^4||\dot{\mathbf{u}}||^2\end{aligned}$$

which implies

$$\frac{1}{2}||\dot{\mathbf{u}}||^2 + \int_\Omega \nu_1|\mathbf{D}\dot{\mathbf{u}}|^2 + \nu_2(\mathbf{D}\mathbf{u} : \mathbf{D}\dot{\mathbf{u}})^2\,d\mathbf{x} \leq c(\nu_1)||\mathbf{D}\mathbf{u}||^4||\dot{\mathbf{u}}||^2.$$

Modulo computability conditions on the initial data, this is sufficient to apply Gronwall's inequality. The key here was to find a lower bound for

$$\int_\Omega (\nu_1 \mathbf{D}\dot{\mathbf{u}} + \nu_2 \mathbf{D}\mathbf{u} : \mathbf{D}\dot{\mathbf{u}}\,\mathbf{D}\mathbf{u}) : \mathbf{D}\dot{\mathbf{u}}\,d\mathbf{x},$$

namely 0. Actually, this result is a consequence of the ellipticity of the associated elliptic operator. Indeed, suppose $\mathbf{T}$ satisfies condition (4.2). Then

$$\begin{aligned}\int_\Omega \frac{d}{dt}T_{ij}(\mathbf{D}\mathbf{u})D_{ij}\dot{\mathbf{u}}\,d\mathbf{x} &= \int_\Omega \partial_{kl}T_{ij}(\mathbf{D}\mathbf{u})D_{kl}\dot{\mathbf{u}}D_{ij}\dot{\mathbf{u}}\,d\mathbf{x} \\ &\geq c(c_1)\int_\Omega |\nabla\dot{\mathbf{u}}|^2 + |\mathbf{D}\mathbf{u}|^{p-2}|\mathbf{D}\dot{\mathbf{u}}|^2\,d\mathbf{x}\end{aligned}$$

which gives an equivalent estimate.

The next natural estimate in the series is to take the derivative of (6.1) with respect to time and then multiply by $\frac{d}{dt}P\partial_j(\nu(\mathbf{Du})D_{ij}\mathbf{u})$ (where P is the L^2 orthogonal projection onto divergence free functions). Integrating this over Ω yields,

$$\int_\Omega \frac{1}{2}\nu(\mathbf{Du})\frac{d}{dt}|\mathbf{D\dot{u}}|^2 + \frac{d}{dt}\nu(\mathbf{Du})\mathbf{Du} : \frac{d}{dt}\mathbf{D\dot{u}}\,d\mathbf{x} + \int_\Omega |P\frac{d}{dt}(\partial_j(\nabla(\nu(\mathbf{Du})D_{ij}\mathbf{u})))|^2 = \\ -\int_\Omega \dot{u}_j\partial_j u_i \frac{d}{dt}P\partial_k(\nu(\mathbf{Du})D_{ik}\mathbf{u}) + u_j\partial_j\dot{u}_i\frac{d}{dt}P\partial_k(\nu(\mathbf{Du})D_{ik}\mathbf{u})\,d\mathbf{x} \quad (6.3)$$

The problem here is the left-hand side, not the right. We would like to express the first integral on the left-hand side as a time derivative of some quantity so as to apply Gronwall's inequality. Inspection shows that because of the nonlinearity it isn't a time derivative. Given the form of the first term, we would perhaps expect to get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\nu(\mathbf{Du})|\mathbf{D\dot{u}}|^2\,d\mathbf{x}$$

which we could make by adding correction terms to (6.3). To do this, though, we would have to find a control for

$$\int_\Omega \frac{d}{dt}\nu(\mathbf{Du})\mathbf{Du} : \frac{d}{dt}\mathbf{D\dot{u}}\,d\mathbf{x} = 2\int_\Omega \nu_2\mathbf{Du} : \mathbf{D\dot{u}}\mathbf{Du} : \frac{d}{dt}\mathbf{D\dot{u}}\,d\mathbf{x}$$

which seems elusive.

One might avoid the problem generated by the nonlinearity by multiplying by $P\partial_k\partial_k\mathbf{u}$ instead of $\frac{d}{dt}P\partial_j(\nu(\mathbf{Du})D_{ij}\mathbf{u})$. But then the problem shifts to the second term on the right hand side. We obtain

$$\frac{d}{dt}\int_\Omega |\nabla\mathbf{\dot{u}}|^2\,d\mathbf{x} + \int_\Omega P\frac{d}{dt}(\partial_j(\nabla(\nu(\mathbf{Du})D_{ij}\mathbf{u})))P(\partial_k\partial_k\dot{u}_i)\,d\mathbf{x} = \\ -\int_\Omega \dot{u}_j\partial_j u_i P(\partial_k\partial_k u_i) + u_j\partial_j\dot{u}_i P(\partial_k\partial_k u_i)\,d\mathbf{x} \quad (6.4)$$

We would like to show

$$\int_\Omega P\frac{d}{dt}(\partial_j(\nabla(\nu(\mathbf{Du})D_{ij}\mathbf{u})))P(\partial_k\partial_k\dot{u}_i)\,d\mathbf{x} \quad (6.5)$$

is positive or is bounded from below by a positive term and a term with controlled growth. This is made difficult by the action of the projection on the product. In the case of the Navier-Stokes equations, however, (6.5) reduces to just

$$\int_\Omega |P\Delta\dot{\mathbf{u}}|^2 \, d\mathbf{x}$$

which is clearly positive and allows a successful application of Gronwall's inequality. Problems like this related the nonlinearity persist for higher derivatives and must be overcome to obtain higher time regularity.

6.3 Higher Space Regularity

Higher space derivatives for nonlinear elliptic systems remain a famous open problem. Because of this notoriety (and thereby implied difficulty), we did not make a serious attempt to tackle higher space regularity. We summarize briefly here starting points for the interested reader. For general elliptic equations, Hölder regularity of the first derivatives can be obtained by the famed de Giorgi theorem, which can be found with exposition in [Gia93]. Further regularity follows from a bootstrap argument. For elliptic systems we have some counterexamples, e.g. [Neč75], to show that we cannot expect everywhere regularity for general elliptic systems. However, these counterexamples do not exclude the possibility that it is possible to prove everywhere regularity for systems with additional structure. In particular, we have the result of Uhlenbeck [Uhl77] that weak solutions of elliptic systems of the form

$$\partial_j \left(F(|\nabla \mathbf{u}|^2) \partial_j u_i \right) = 0$$

have Hölder continuous first derivatives. Notice the similarity between this system and the Poisson-like system considered in Chapter 3. Finally, we remark that in two dimensions the regularity problems are not the same. Indeed, there is a very recent preprint [KMS97] which proves that in two dimensions the stationary version of (6.1) with periodic boundary conditions has classical solutions.

6.4 Final Words

Despite what we are unable to prove, we must keep in mind what has been accomplished. We have shown a new method for obtain interior regularity of elliptic systems with solenoidal constraint. Moreover, we have extended the class of systems for which this is possible. We have shown how interior regularity for the elliptic system allows us to prove (and clarify) interior regularity for the model parabolic system of interest. We have done all this using straight-forward, classically motivated, techniques. Therefore, our failure to find classical solutions should not be viewed too harshly. Our hope is that by presenting our shortcomings the reader might be inclined to think about this problem and help drive our knowledge toward the truth.

Bibliography

[Ama94] H. Amann, *Stability of the rest state of a viscous incompressible fluid*, Arch. Rat. Mech. Anal. **126** (1994), 231–242.

[BBN94] H. Bellout, F. Bloom, and J. Nečas, *Young measure-valued solutions for non-newtonian incompressable fluids*, Comm. in PDE **19** (1994), 1763–1803.

[Gal94] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, vol. 1, Springer-Verlag, 1994.

[Gia93] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems*, Birkhauser, Basel, 1993.

[GM82] M. Giaquinta and G. Modica, *On linear systems of the type of the stationary navier-stokes system*, J. fur Math. **330** (1982), 173–214.

[Hey76] J. G. Heywood, *On uniqueness questions in the theory of viscous flow*, Acta Math. **136** (1976), 61–102.

[Hey80] J.G. Heywood, *The navier-stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **5** (1980), 639–681.

[Hey93] J.G. Heywood, *Remarks on the possible global regularity of solutions of the three-dimensional navier-stokes equations*, Progress in Theoretical and Computational Fluid Mechanics, Pitman Research Notes in Mathematics, no. 308, Longman Group, 1993.

[KMS97] P. Kaplický, J. Málek, and J. Stará, *Full regularity of weak solutions ot a class of nonlinear fluids in two dimensions – stationary, periodic problems*, submitted to Comment. Math. Carol. Univ., 1997.

[Lad69] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, 2 ed., Gordon and Breach, New York, 1969.

[LU68] O.A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[Max97] D. Maxwell, *On an elliptic regularity theorem with applications to a class of non-newtownian fluids*, to appear, 1997.

[MNR93] J. Málek, J. Nečas, and M. Růžička, *On the non-newtonian incompressible fluids*, Meth. Math. Model. App. Sci. **3** (1993), 35–63.

[MNR96] J. Málek, J. Nečas, and M. Růžička, *On weak solutions ot a class of non-newtonian incompressible fluids bounded three-dimensional domains. the case* $p \geq 2$, submitted to Adv. in Diff. Eq., Preprint SFB 256, No. 481, 1996.

[MNRR96] J. Málek, J. Nečas, M. Rokyta, and M. Růžička, *Weak and measure-valued solutions to evolutionary partial differential equations*, Applied Mathematics and Mathematical Computation, vol. 13, Chapman and Hall, 1996.

[Neč66] J. Nečas, *Sur les normes équivalentes dans $W_p^k(\Omega)$ et sur la coercivité des formes formellement positives*, Les presses de l'Université de Montréal, 1966.

[Neč75] J. Nečas, *Example of an irregular solutiosn ot a nonlinear elliptic system with analytic coefficients and conditions for regularity*, Theory of Non-Linear Operators, Abhandluing Akad. der Wissen. der DDR, Proc. of a Summer School held in Berlin, 1975.

[Neč83] J. Nečas, *Introduction to the theory of nonlinear elliptic equations*, Teubner-Texte zur Mathematik, no. 52, Teubner Verlagsgesellschaft, 1983.

[Sma63] J. S. Smagorinski, *General circularion model of the atmosphere*, Mon. Weather Rev. **91** (1963), 99–164.

[SŠ73] V.A. Solonnikov and V.E. Ščadilov, *On a boundary value problem for a stationary system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov **125** (1973), 186–199.

[Uhl77] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), 219–240.

Appendix A
A Direct Coercivity Calculation

We wish to show directly that for two functions $\mathbf{u}$ and $\mathbf{v}$ in $W_0^{1,4}(\Omega)$,

$$\int_\Omega \left((\nu_1 + \nu_2|\nabla \mathbf{u}|^2)\nabla_{ij}\mathbf{u} - (\nu_1 + \nu_2|\nabla \mathbf{v}|^2)\nabla_{ij}\mathbf{v}\right) \nabla_{ij}(\mathbf{u} - \mathbf{v})\, d\mathbf{x} \geq c||\nabla(\mathbf{u} - \mathbf{v})||_4^4.$$

This calculation inspired Corollary 4.1 and is very easy to follow. Thus this calculation is included here for both completeness and interest.

Lemma A.1 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be in* $W_0^{1,4}(\Omega)$*,*

$$\int_\Omega \left((\nu_1 + \nu_2|\nabla \mathbf{u}|^2)\nabla_{ij}\mathbf{u} - (\nu_1 + \nu_2|\nabla \mathbf{v}|^2)\nabla_{ij}\mathbf{v}\right) \nabla_{ij}(\mathbf{u} - \mathbf{v})\, d\mathbf{x} \geq \frac{\nu_2}{4}||\nabla(\mathbf{u} - \mathbf{v})||_4^4.$$

Proof:

Let $\mathbf{u}(\nabla \mathbf{u})$ denote $(\nu_1 + \nu_2|\nabla \mathbf{u}|^2)$, let $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and let $\nabla \mathbf{u} : \nabla \mathbf{v}$ denote $\nabla_{ij}\mathbf{u}\nabla_{ij}\mathbf{v}$. Then

$$\int_\Omega \nu(\nabla \mathbf{u})\nabla \mathbf{u} : \nabla(\mathbf{w}) - \nu(\nabla \mathbf{v})\nabla \mathbf{v} : \nabla \mathbf{w}\, d\mathbf{x} = \tag{A.1}$$

$$\begin{aligned}
&= \int_\Omega \left(\nu(\nabla \mathbf{u}) - \nu(\nabla \mathbf{v})\right) \nabla \mathbf{u} : \nabla \mathbf{w} + \nu(\nabla \mathbf{v})\nabla \mathbf{w} : \nabla \mathbf{w}\, d\mathbf{x} \\
&= \int_\Omega \nu_2 \left(|\nabla \mathbf{u}|^2 - |\nabla \mathbf{v}|^2\right) \nabla \mathbf{u} : \nabla \mathbf{w} + \nu(\nabla \mathbf{v})|\nabla \mathbf{w}|^2\, d\mathbf{x} \\
&= \int_\Omega \nu_2 \nabla(\mathbf{u} + \mathbf{v}) : \nabla \mathbf{w} \nabla \mathbf{u} : \nabla \mathbf{w} + \nu(\nabla \mathbf{v})|\nabla \mathbf{w}|^2\, d\mathbf{x} \\
&= \int_\Omega \nu_2(\nabla \mathbf{u} : \nabla \mathbf{w})^2 + \nu_2(\nabla \mathbf{u} : \nabla \mathbf{w})(\nabla \mathbf{v} : \nabla \mathbf{w}) + \mathbf{u}(\nabla \mathbf{v})|\nabla \mathbf{w}|^2.
\end{aligned} \tag{A.2}$$

Now, noticing that expression $\int_\Omega \nu(\nabla\mathbf{u})\nabla\mathbf{u} : \nabla(\mathbf{u}-\mathbf{v}) - \nu(\nabla\mathbf{v})\nabla\mathbf{v} : \nabla(\mathbf{u}-\mathbf{v})\,d\mathbf{x}$ is symmetric in $\mathbf{u}$ and $\mathbf{v}$, since

$$\int_\Omega \nu(\nabla\mathbf{u})\nabla\mathbf{u} : \nabla(\mathbf{u}-\mathbf{v}) - \nu(\nabla\mathbf{v})\nabla\mathbf{v} : \nabla(\mathbf{u}-\mathbf{v})\,d\mathbf{x} = \int_\Omega \nu(\nabla\mathbf{u})|\nabla\mathbf{u}|^2 + \nu(\nabla\mathbf{v})|\nabla\mathbf{v}|^2 - (\nu(\nabla\mathbf{u}) + \nu(\nabla\mathbf{v}))\nabla\mathbf{u} : \nabla\mathbf{v}\,d\mathbf{x},$$

we conclude from (A.2) that

$$\int_\Omega \nu(\nabla\mathbf{u})\nabla\mathbf{u} : \nabla(\mathbf{u}-\mathbf{v}) - \nu(\nabla\mathbf{v})\nabla\mathbf{v} : \nabla(\mathbf{u}-\mathbf{v})\,d\mathbf{x} = \int_\Omega \nu_2(\nabla\mathbf{v} : \nabla\mathbf{w})^2 + \nu_2(\nabla\mathbf{u} : \nabla\mathbf{w})(\nabla\mathbf{v} : \nabla\mathbf{w}) + \mathbf{u}(\nabla\mathbf{u})|\nabla\mathbf{w}|^2. \quad \text{(A.3)}$$

Averaging (A.2) and (A.3) we see then that

$$\begin{aligned}
\int_\Omega \nu(\nabla\mathbf{u})\nabla\mathbf{u} : \nabla\mathbf{w} - \nu(\nabla\mathbf{v})\nabla\mathbf{v} : \nabla\mathbf{w}\,d\mathbf{x} &= \\
&= \frac{\nu_2}{2}\int_\Omega (\nabla\mathbf{v} : \nabla\mathbf{w})^2 + 2(\nabla\mathbf{v} : \nabla\mathbf{w})(\nabla\mathbf{u} : \nabla\mathbf{w}) + (\nabla\mathbf{u} : \nabla\mathbf{w})^2\,d\mathbf{x} + \\
&\quad + \int_\Omega \left(\nu_1 + \frac{\nu_2}{4}\left(|\nabla\mathbf{v}|^2 + |\nabla\mathbf{u}|^2\right)\right)|\nabla\mathbf{w}|^2\,d\mathbf{x} \\
&= \int_\Omega \nu_2\left(\nabla(\mathbf{u}+\mathbf{v}) : \nabla\mathbf{w}\right)^2\,d\mathbf{x} + \int_\Omega \left(\nu_1 + \frac{\nu_2}{4}\left(|\nabla\mathbf{v}|^2 + |\nabla\mathbf{u}|^2\right)\right)|\nabla\mathbf{w}|^2\,d\mathbf{x} \\
&\geq \int_\Omega (\nu_1 + \frac{\nu_2}{4}|\nabla\mathbf{w}|^2)|\nabla\mathbf{w}|^2\,d\mathbf{x} \\
&\geq \frac{\nu_2}{4}||\nabla\mathbf{w}||_4^4. \qquad \text{(A.4)}
\end{aligned}$$

□

Since nothing in the previous proof relied on the fact that we used $\nabla\mathbf{u}$ as opposed to $\mathbf{Du}$, we obtain a similar result in this case.

Lemma A.2 *Let $\mathbf{u}$ and $\mathbf{v}$ be in $J_0^{1,4}(\Omega)$. Then,*

$$\int_\Omega \left((\nu_1 + \nu_2|\mathbf{Du}|^2)D_{ij}\mathbf{u} - (\nu_1 + \nu_2|\mathbf{Dv}|^2)D_{ij}\mathbf{v}\right) D_{ij}(\mathbf{u}-\mathbf{v})\,d\mathbf{x} \geq \frac{\nu_2}{4}||\mathbf{D}(\mathbf{u}-\mathbf{v})||_4^4.$$