Initial Data for Black Holes and Rough Spacetimes

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Abstract

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We construct two new classes of solutions of the Einstein constraint equations. First, we construct a family of solutions on asymptotically Euclidean manifolds with boundary such that the boundary is an apparent horizon. This initial data has application to the general relativistic N-body problem. Second, we construct a family of low regularity asymptotically Euclidean solutions of the constraint equations. These solutions have a metric in H^s_{loc} with $s > 3/2$ and are required for existence theorems for rough solutions of the Einstein evolution equations. In both cases, we adapt and extend the constant mean curvature conformal method to apply it to these new settings.

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DEDICATION

For my parents.

Chapter 1

INTRODUCTION

Einstein's general theory of relativity [Ei15] is a geometric model of space and time. It describes the universe as a Lorentzian 4-manifold $(\mathcal{M}, \mathbf{g})$ on which gravity, encoded by the metric, and matter interact via a system of PDEs known as the Einstein equations. Hidden in the geometric formulation lies an initial value problem reflecting the theory's origins in classical dynamics. One significant difference between the initial value problem of general relativity and its classical counterpart is the complexity of constructing suitable initial data. Data for the initial value problem cannot be freely specified, but must itself satisfy a system of PDEs known as the Einstein constraint equations. So for every application of the initial value problem, there is the associated problem of finding initial data suitable for that application.

Our goal is to construct solutions of constraint equations for two applications of current interest. The first has a geometric character and concerns the construction of initial data suitable for modeling an isolated system of black holes. The second is more analytic in nature and is related to well-posedness questions for the Einstein equations. Before describing these problems more explicitly, we start by reviewing some aspects of the initial value problem.

1.1 The Einstein Equations and the Einstein Constraint Equations

A Lorentzian manifold is a smooth topological manifold $\mathcal{M}^n$ equipped with a metric $\mathbf{g}$ having signature $(-, +, \cdots, +)$. Like other field equations in physics, the Einstein equations arise from a variational principle. We require the action

$$\int_M R_{\mathbf{g}} + L \, dV_{\mathbf{g}} \tag{1.1}$$

be stable under compact perturbations of the metric. In (1.1), $R_{\mathbf{g}}$ is the scalar curvature of $\mathbf{g}$ and L is the Lagrangian associated with non-gravitational fields. When $L = 0$, the integral (1.1) is known as the Einstein-Hilbert action. Computing the variation of (1.1) with respect to $\mathbf{g}$ we obtain the field equations

$$\mathrm{Ric}_{\mathbf{g}} - \frac{1}{2} R_{\mathbf{g}} \, \mathbf{g} = T. \tag{1.2}$$

The tensor T is the variation of $\int_M L \, dV_{\mathbf{g}}$ with respect to $\mathbf{g}$ and is known as the stress-energy tensor. A vacuum spacetime is a Lorentzian manifold $(\mathcal{M}^4, \mathbf{g})$ that satisfies (1.2) with $T = 0$. In this case, we can take the trace of (1.2) to find $R_{\mathbf{g}} = 0$ and obtain the vacuum Einstein equations

$$\mathrm{Ric}_{\mathbf{g}} = 0. \tag{1.3}$$

The metric $\mathbf{g}$ naturally partitions each tangent space into three regions. We say a vector X is timelike, spacelike or null if $\mathbf{g}(X, X)$ is negative, positive or zero respectively, and we say a vector is causal if it is either timelike or null. A curve is timelike, spacelike, or null if it has a timelike, spacelike, or null tangent vector at each point. A hypersurface M of $\mathcal{M}$ is timelike, spacelike, or null if its tangent space at each point has a normal vector that is spacelike, timelike, or null respectively. There is a special class of spacelike hypersurfaces that arise in the initial value problem. A Cauchy surface of $\mathcal{M}$ is a spacelike hypersurface having the property that every inextendible timelike curve in $\mathcal{M}$ intersects M once and only once. Not every Lorentzian manifold admits a Cauchy surface; those that do are called globally hyperbolic. Every globally hyperbolic Lorentzian manifold admits a continuous, globally defined, timelike vector field F. The choice of such a vector field is called a time orientation, and any other timelike vector X is said to be future or past pointing if $\mathbf{g}(F, X)$ is negative or positive respectively. We will tacitly assume that all Lorentzian manifolds are globally hyperbolic and have been given a time orientation.

Suppose $(\mathcal{M}, g)$ is a globally hyperbolic vacuum spacetime with Cauchy surface M. Let g be the Riemannian metric on M induced by $\mathbf{g}$, let n be the future pointing timelike unit normal to M, and let K denote the extrinsic curvature of M computed with respect to n, i.e. $K(X, Y) = -\left\langle \nabla^{\mathbf{g}}_X n, Y \right\rangle$ where $\nabla^{\mathbf{g}}$ is the connection on $\mathcal{M}$. The Gauss-Codazzi equations permit the computation of $n \lrcorner$

$(\mathrm{Ric}_{\mathbf{g}} - \frac{R}{2}\mathbf{g})$ in terms of g and K. Since $\mathrm{Ric}_{\mathbf{g}} = 0$ we obtain

$$R - |K|^2 + \mathrm{tr}\, K^2 \quad = 0 \tag{1.4}$$

$$\mathrm{div}\, K - d\ \mathrm{tr}\, K \quad = 0. \tag{1.5}$$

where all quantities of (1.4) and (1.5) involving a metric are computed with respect to g. Equations (1.4) and (1.5) are known as the Hamiltonian and momentum constraint equations respectively. Together they form the Einstein constraint equations.

The vacuum initial value problem of general relativity is the following. Given initial data (M, g, K), find a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ satisfying the vacuum Einstein equations and an embedding $\iota : M \mapsto \mathcal{M}$ such that $\iota(M)$ is a Cauchy surface for $\mathcal{M}$ and such that $\mathbf{g}$ induces g and K on $\iota(M)$. A spacetime $\mathcal{M}$ satisfying these properties is called a Cauchy development of (M, g, K) and is by definition a globally hyperbolic Lorentzian manifold. If it is also true that every Cauchy development of (M, g, K) can be isometrically embedded in $\mathcal{M}$, we say $\mathcal{M}$ is called the maximal development of (M, g, K) (one can show a maximal development is unique up to isomorphism). As we have seen, (M, g, K) must satisfy the constraint equations in order to have a Cauchy development. The following theorem shows that this condition is also sufficient.

Theorem 1.1 [CBG69] *Given smooth initial data (M, g, K) satisfying the constraint equations* (1.4) *and* (1.5)*, there exists a smooth, maximal, globally hyperbolic Cauchy development of the initial data.*

As a consequence, there is a strong connection between globally hyperbolic solutions of the Einstein equations and solutions of the Einstein constraint equations. However, two distinct solutions of the constraint equations can generate isometric maximal developments. One cause of this phenomenon arises from gauge freedom associated with the diffeomorphism group. If (M, g, K) is a solution of the Einstein constraint equations, and if $\Phi : M \to M$ is a diffeomorphism, then $(M, \Phi^* g, \Phi^* K)$ is also a solution of the constraint equations. More significantly, a globally hyperbolic manifold can be foliated with Cauchy surfaces, and each such surface determines a solution of the constraint equations. So it is not the case that solutions of the constraint equations and maximal developments of the Einstein equations are in one-to-one correspondence. Never-the-less, it is important to understand the set of solutions of the constraint equations as a first step to constructing maximal developments.

The Hamiltonian constraint (1.4) is a scalar equation and the momentum constraint (1.5) is a vector equation; together they constitute 4 restrictions on 12 unknowns. So the constraint equations constitute an underdetermined system, and the resulting degrees of freedom reflect the ability to choose different initial conditions. The central questions about the constraint equations fall broadly into two categories.

1. Can we parameterize the full set of solutions on a manifold M, or at least an interesting subset?

2. Given a property $\mathcal{P}$ of initial data, can we construct solutions of the constraints with property $\mathcal{P}$?

The full answer to the first question is currently out of reach. But there is a general technique, known as the conformal method, for constructing a large class of solutions. Our strategy is to extend and adapt the conformal method to address two problems of the second type, one concerning black hole initial data and one concerning low-regularity solutions.

1.2 The Conformal Method

The conformal method of Lichnerowicz [Li44] and Choquet-Bruhat and York [CBY80] starts by decomposing initial data $(M^3, \tilde{g}, \tilde{K})$ into degrees of freedom that can be freely specified and degrees of freedom that are to be found by solving a determined system of PDEs. The free parameters are a representative Riemannian metric g of a conformal class $[g]$, a trace-free symmetric $(0,2)$-tensor S, and a function τ. We will call the set (M, g, S, τ) the conformal data. The variables to be determined are a conformal factor ϕ and vector field W. We reconstruct the metric and second fundamental form via

$$\tilde{g} = \phi^4 g$$
$$\tilde{K} = \phi^{-2}\left(S + \mathbb{L}W\right) + \frac{\tau}{3}\tilde{g}$$

where $\mathbb{L}$ is the conformal Killing operator. We recall that on a 3-manifold,

$$\mathbb{L}W = \mathcal{L}_W g - \frac{2}{3}\operatorname{div} W g$$

where $\mathcal{L}$ is the Lie derivative. Now $\mathbb{L}W$ is a trace-free symmetric $(0,2)$-tensor, and therefore so is $S+\mathbb{L}W$. An easy computation shows that if S is any trace-free symmetric $(0,2)$-tensor, then

$$\operatorname{div}_{\tilde{g}}\, \phi^{-2}S = \phi^{-6}\operatorname{div}_g S.$$

Using this relationship and the familiar rule for the change of scalar curvature under conformal transformations

$$R_{\tilde{g}} = \phi^{-5}\left(-8\Delta_{\,g}\phi + R_g\phi\right)$$

we can write the vacuum constraint equations for $(M,\tilde{g},\tilde{K})$ in terms of ϕ, W, and the conformal data (M,g,S,τ). We find

$$\begin{aligned} -8\Delta_{\,g}\phi + R_g\phi &= \left|S+\mathbb{L}W\right|_g^2\phi^{-7} - \frac{2}{3}\tau^2\phi^5 \\ \Delta_{\mathbb{L}_g}W &= \frac{2}{3}d\tau\,\phi^6 - \operatorname{div}_g S \end{aligned} \tag{1.6}$$

where $\Delta_{\mathbb{L}_g}$ is the vector Laplacian $\operatorname{div}_g \mathbb{L}_g$. From now on we will assume, unless stated otherwise, that quantities involving a metric are computed with respect to g.

The question of solving the constraint equations becomes one of finding conformal data for which system (1.6) admits a solution. Little is known about this question without making additional assumptions. The most satisfactory results come from the hypothesis τ is constant. Since $\operatorname{tr}_{\tilde{g}}\tilde{K} = \tau$, the resulting solutions are known as constant mean curvature (CMC) solutions. Under the CMC hypothesis, the system (1.6) decouples and we find

$$\begin{aligned} -8\Delta\,\phi + R\phi &= \left|S+\mathbb{L}W\right|^2\phi^{-7} - \frac{2}{3}\tau^2\phi^5 \\ \Delta_{\mathbb{L}}W &= -\operatorname{div} S. \end{aligned} \tag{1.7}$$

Although we will not work with compact manifolds, it is insightful to start by considering system (1.7) on a compact manifold.

The equation for W

$$\Delta_{\mathbb{L}}W = -\operatorname{div} S \tag{1.8}$$

is linear and has a straightforward analysis. An easy computation of the symbol of $\Delta_{\mathbb{L}}$ shows this operator is elliptic. Since the formal adjoint of $\mathbb{L}$ is $-\operatorname{div}$, we have $\Delta_{\mathbb{L}} = -\mathbb{L}^*\mathbb{L}$ is self adjoint and hence is Fredholm with index 0. In particular, we can solve (1.8) if and only if

$$\int_M \langle X, \operatorname{div} S\rangle\; dV = 0$$

for every $X \in \ker \Delta_{\mathbb{L}}$. We note that if $X \in \ker \Delta_{\mathbb{L}}$, then integrating by parts we have

$$0 = \int_M \langle \Delta_{\mathbb{L}} X, X \rangle \ dV = - \int_M \langle \mathbb{L}X, \mathbb{L}X \rangle \ dV$$

and hence X is a conformal Killing field. But then another integration by parts argument shows

$$\int_M \langle X, \operatorname{div} S \rangle \ dV = - \int_M \langle \mathbb{L}X, S \rangle \ dV = 0.$$

So there is always a solution W of (1.8), and it is unique up to the addition of a conformal Killing field X. Setting $\sigma = S + \mathbb{L}W$ we see that σ is a symmetric, trace-free, and divergence free $(0,2)$-tensor. We will call such tensors transverse-traceless and we note that σ is independent of the solution W of (1.8). So specifying conformal data (M, g, S, τ) is equivalent to specifying conformal data (M, g, σ, τ) where σ is a transverse-traceless tensor. In terms of this latter form of the conformal data, the first equation of (1.7) becomes

$$-8\Delta\, \phi + R\phi = |\sigma|^2\, \phi^{-7} - \frac{2}{3}\tau^2 \phi^5, \tag{1.9}$$

which is known as the Lichnerowicz equation.

The analysis of the Lichnerowicz equation has an interesting connection with the Yamabe problem of finding a metric in a conformal class having constant scalar curvature. For example, when $\sigma \equiv 0$, solving (1.9) is equivalent to finding a metric conformally related to g with constant scalar curvature $-\frac{2}{3}\tau^2$. We recall the Yamabe invariant λ_g of a compact 3-manifold (M^3, g) is defined by

$$\lambda_g = \inf_{f \in C^\infty(M), f \not\equiv 0} \frac{\int_M 8\,|\nabla f|^2 + Rf^2 \ dV}{||f||^2_{L^6}}.$$

The solution to the Yamabe problem (see, e.g. the review article [LP87]) shows that (M, g) is conformally related to a metric with constant scalar curvature R if and only if λ_g has the same sign as R. Although the problem of solving the Lichnerowicz equation is not as deep as that of the Yamabe problem, the Yamabe invariant plays a central role in understanding when (1.9) admits a solution. We have from [Is95], building on previous work [CB72] [OY73], that (1.9) has a unique positive solution ϕ in the following cases.

	$\sigma \equiv 0, \tau = 0$	$\sigma \not\equiv 0, \tau = 0$	$\sigma \equiv 0, \tau \neq 0$	$\sigma \not\equiv 0, \tau \neq 0$
$\lambda_g > 0$	No	Yes	No	Yes
$\lambda_g = 0$	Yes	No	No	Yes
$\lambda_g < 0$	No	No	Yes	Yes

Hence the conformal method leads to a construction of all CMC solutions of the constraint equations on a compact manifold.

Our constructions involve an important class of non-compact Riemannian manifolds where the conformal method also yields a construction of all CMC solutions of the constraints. These are the asymptotically Euclidean (AE) manifolds used for modeling isolated gravitational systems. Loosely speaking, an AE manifold has a compact core and a number of ends on which the metric is asymptotic to the Euclidean metric at far distances. An AE solution $(M, \tilde{g}, \tilde{K})$ of the constraints has the asymptotics on each end

$$\tilde{g} = \overline{g} + o(1)$$
$$\tilde{K} = o(r^{-1}),$$

where $\overline{g}$ is the Euclidean metric. We make these decay conditions precise in Chapter 3 using weighted Sobolev spaces.

The approach of the conformal method in the AE setting follows that for compact manifolds. Equation (1.8), used to construct transverse traceless tensors, is in some ways easier to analyse on AE manifolds. It can be shown that there are no conformal Killing fields vanishing at infinity, so the vector Laplacian is always an isomorphism on suitable function spaces. To study the Lichnerowicz equation, we start with conformal data (M, g, σ, τ) and wish to find a conformally related solution $(M, \tilde{g}, \tilde{K})$ of the constraint equations. From the CMC hypothesis and the decay of $\tilde{K}$ at infinity, we have $\tau = \operatorname{tr}_{\tilde{g}} \tilde{K} = 0$; such solutions are called maximal and are the Lorentzian analogues of minimal surfaces. The Hamiltonian constraint for a maximal solution $(M, \tilde{g}, \tilde{K})$ reads

$$R_{\tilde{g}} = \left|\tilde{K}\right|_{\tilde{g}}^2.$$

We conclude that if (1.9) has a solution for (M, g, σ), then (M, g) must be conformally equivalent to an AE metric with non-negative scalar curvature. In fact, something stronger is true. It was shown by Cantor in [Ca79a] that (1.9) is solvable if and only if (M, g) is conformally related to a scalar flat metric. Now if (M, g) is conformally related to a scalar flat metric $(M, \hat{g})$, we have

$$\int_M 8\left|\nabla^{\hat{g}} f\right|^2_{\hat{g}} + R_{\hat{g}} f^2 \, dV_{\hat{g}} = \int_M 8\left|\nabla^{\hat{g}} f\right|^2_{\hat{g}} \, dV_{\hat{g}} > 0 \tag{1.10}$$

for all smooth, compactly supported functions f not identically 0. For fixed f, the sign of first integral in (1.10) is a conformal invariant. So if (M, g) is conformally equivalent to a scalar flat metric, then

$$\int_M 8\left|\nabla f\right|^2 + R f^2 \, dV > 0 \quad \text{for all } f \in C^\infty_{\mathrm{c}}(M),\ f \not\equiv 0 \tag{1.11}$$

as well. It was previously reported [CaB81] [CBIY00] that (1.11) is also a sufficient condition. This is not quite true, and we have shown [Ma03] that the correct condition is

$$\lambda_g = \inf_{f \in C^\infty_{\mathrm{c}}(M), f \not\equiv 0} \frac{\int_M 8\left|\nabla f\right|^2 + R f^2 \, dV}{||f||^2_{L^6}} > 0. \tag{1.12}$$

The proof that (1.9) is solvable if and only if $\lambda_g > 0$ uses the method of sub- and super-solutions to solve the semilinear Lichnerowicz equation. The argument in the end is quite simple. If $\lambda_g > 0$, we can make a conformal change to a scalar flat metric. Letting $\phi = 1 + v$, the Lichnerowicz equation becomes

$$-8\Delta\, v = \left|\sigma\right|^2 (1 + v)^{-7}. \tag{1.13}$$

Now $v_- = 0$ is a subsolution of (1.13) (i.e $-8\Delta\, v_- \le \left|\sigma\right|^2 (1 + v_-)^{-7}$). Moreover, letting v_+ be the solution of

$$-8\Delta\, v_+ = \left|\sigma\right|^2$$

that decays at infinity, it follows from a maximum principle argument that $v_+ \ge 0$. Since

$$\left|\sigma\right|^2 (1 + v_+)^{-7} \le \left|\sigma\right|^2$$

it follows that v_+ is a non-negative supersolution of (1.13). Since $v_- \le v_+$, a barrier argument then shows there exists a solution of (1.13).

In summary, the CMC-conformal method leads to a decoupling of the momentum and Hamiltonian constraint equations. The momentum constraint is linear and is addressed by understanding the mapping properties of the vector Laplacian. The Hamiltonian constraint is semilinear and is solved using a barrier argument. The first step is to make a conformal change to make the equation easier to analyze, and the second step is to find a sub/super-solution pair. This is the programme we follow to construct initial data in our applications.

1.3 Black Hole Initial Data

The black hole region $\mathcal{B}$ of a spacetime $(\mathcal{M}, \mathbf{g})$ is the set from which light cannot escape to infinity. This notion can be made precise if $(\mathcal{M}, \mathbf{g})$ possesses an additional structure known as conformal infinity (see Chapter 2). Although this definition of a black hole is sufficient for proving theorems, it suffers from the drawback that in general one cannot know if a point $p \in \mathcal{B}$ unless one also knows the entire causal future of p. In particular, given a Cauchy surface M of $\mathcal{M}$, there is no known way of determining $\mathcal{B} \cap M$ without generating its future development. This is a significant obstacle to constructing initial data for black hole spacetimes.

Fortunately, there exist structures called apparent horizons and trapped surfaces that can be detected in initial data and accurately predict the development of a black hole in the spacetime. If Σ is an apparent horizon in M, then it is guaranteed that $\Sigma \subset M \cap \mathcal{B}$. Heuristically, one can create initial data for N black holes by creating initial data with N apparent horizons $\{\Sigma_i\}_{i=1}^N$. There is no assurance, however, that Σ_i and Σ_j will be contained in separate components of $\mathcal{B} \cap M$, so it is not certain that a spacetime with N apparent horizons really does contain N black holes. We can only presume this is is so if the horizons are well separated.

Traditional methods of generating initial data containing trapped surfaces and apparent horizons do so indirectly by working with manifolds with nontrivial topology [Mi63], [BL63], [YB80]; these methods guarantee the existence of an apparent horizon somewhere in the data but typically do not dictate precisely where. A direct approach to the problem, first proposed for numerical study in [Th87], is to work with a manifold with boundary and specify that the boundary be an apparent horizon. Until recently, however, there had not been a rigorous construction of such initial data.

We present in this dissertation our constuction from [Ma03] of asymptotically Euclidean solutions of the constraint equations on a manifold with boundary such that each component of the boundary is an apparent horizon. Contemporaneous work by Dain in [Da03] treated a similar problem, solving the constraints with a trapped surface boundary condition. The two works used similar methods, but the resulting boundary conditions are somewhat different. In fact, it remains an open problem to connect the two works in a larger framework; we discuss this and other open problems in Section 6.1.2.

The apparent horizon boundary condition can be written as

$$-\operatorname{tr} K + K(\nu,\nu) - H = 0 \quad \text{on } \partial M, \tag{1.14}$$

where ν is the exterior unit normal to the boundary and H is the mean curvature of ∂M computed with respect to the interior unit normal. Applying the CMC conformal method, equation (1.14) becomes a boundary condition for Lichnerowicz equation. We obtain

$$\begin{aligned} -8\Delta\,\phi + R\phi - |\sigma|^2\,\phi^{-7} &= 0 \\ 4\partial_\nu\phi + H\phi - \sigma(\nu,\nu)\phi^{-3} &= 0 \quad \text{on } \partial M \end{aligned} \tag{1.15}$$

where σ is transverse-traceless tensor. Our main results for the apparent horizon boundary value problem provide sufficient conditions on the conformal data (M, g, σ) under which system (1.15) is solvable and show that there is a large class of conformal data that satisfy these conditions.

1.4 Rough Initial Data

By making a suitable choice of coordinates, known as harmonic or wave coordinates, the vacuum Einstein equations for the Lorentzian metric $\mathbf{g}$ take the form

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\mathbf{g}_{\mu\nu} = N_{\mu\nu}(\mathbf{g}, \partial\mathbf{g})$$

where $N_{\mu\nu}$ is quadratic in the derivatives of $\mathbf{g}$. This is a nonlinear hyperbolic system of PDEs, and it is natural to determine function spaces in which the evolution problem is well posed. The first local well-posedness results for the Einstein equations were established in [FB52] for initial data $(g, K) \in C^5 \times C^4$. Subsequent improvements lead to a well-posedness result [HKM77] for the Einstein equations that requires initial data with $(\partial g, K) \in H^{s-1} \times H^{s-1}$ with $s > 5/2$ (by ∂g here we mean the derivatives of g in suitable coordinates). We will call a solution with this last level of regularity a classical solution. Recent work in the theory of nonlinear hyperbolic PDEs has lowered the amount of regularity required. Smith and Tataru [ST] have obtained local well-posedness for nonlinear wave equations with initial data in $H^s \times H^{s-1}$ with $s > 2$. In the case of the vacuum Einstein equations, Klainerman and Rodnianski [KR] established an a-priori estimate for the time of existence of a classical solution in terms of the norm of $(\partial g, K)$ in (H^{s-1}, H^{s-1}), again with $s > 2$. These results should have lead to an existence theorem for rough solutions, but the corresponding low regularity theory of the constraint equations was not sufficiently well developed. It was not known if there existed any rough solutions of the constraint equations. Moreover, to pass from the a-priori estimate in [KR] to an existence theorem for rough initial data requires the existence of a sequence of classical solutions of the constraints approximating the rough solution; this approximation theorem was also missing.

We present here our proof from [Ma04] that the conformal method can be extended to construct suitable rough solutions of the constraints, and that these solutions can be approximated in an appropriate topology by classical solutions. Specifically, we generate asymptotically Euclidean solutions in $H^s_{\rm loc} \times H^{s-1}_{\rm loc}$ with $s > 3/2$. To compare the lower bound $s > 3/2$ with previous results, we must keep in mind that the constraint equations have typically been solved either in Hölder spaces or in Sobolev spaces $W^{k,p}_{\rm loc} \times W^{k-1,p}_{\rm loc}$, where k is an integer. The classical lower bound [CBIY00] for the existence of solutions of the constraint equations was $k > 3/p + 1$. These metrics have one continuous derivative and can be thought of as analogous to metrics in $H^s_{\rm loc}$ with $s > 5/2$. This lower bound was improved in the settings of compact manifolds [CB03] and asymptotically Euclidean manifolds [Ma03] to $k \geq 2$ and $k > 3/p$. The restriction $k > 3/p$ ensures that the metric is continuous, while the inequality $k \geq 2$ further implies that the curvature belongs to $L^p_{\rm loc}$. Taking $k = 2$ and $p = 2$, these results provide for $H^2_{\rm loc} \times H^1_{\rm loc}$ solutions of the constraint equations. But they do not construct solutions directly in the spaces of interest ($H^s_{\rm loc} \times H^{s-1}_{\rm loc}$ with $s > 2$), nor do they provide an approximation theorem.

Now $s = 3/2$ is the scaling limit for the Einstein equations, so this is a natural lower bound for local well-posedness results. It has been suggested [KR03] that it might not be possible to obtain local well-posedness down to $s > 3/2$. In the case of the constraint equations, however, we have shown that working in spaces with $s > 3/2$ is feasible. In fact, the restriction $s > 3/2$ is analogous to the condition $k > 3/p$ from [CB03] [Ma03] as these thresholds ensure the metric is continuous. A novel feature of the solutions considered here is that when $3/2 < s < 2$, the curvature of g is in general only a distribution, not necessarily an integrable function.

1.5 Strategy

Rather than treat each application separately, our approach is to prove results for both settings in parallel. The steps for the conformal method are similar in both applications and we wish to avoid having two similar copies of each theorem. We begin in Chapter 2 by motivating the apparent horizon condition and laying out the programme for adapting the conformal method to this problem. Chapter 3 then starts the mathematical analysis with results for our choice of weighted function spaces. These spaces were introduced some time ago by Triebel [Tr76a], but are not well-known and are an important piece of our low regularity constructions. We then establish in Chapter 4 a priori estimates for the Laplacian and vector Laplacian of a rough metric. Using these estimates, we prove in Chapter 5 the propositions needed to apply the conformal method to the apparent horizon boundary problem and to the low regularity problem. Finally, Chapter 6 contains the constructions of solutions for both applications.

1.6 Notation

The set of tempered distributions on $\mathbb{R}^n$ is $\mathcal{S}^*$. We use the Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ of functions with k derivatives in $L^p(\mathbb{R}^n)$, $H^s(\mathbb{R}^n)$ defined via Fourier transforms, and their cousins $W^{k,p}(\Omega)$ and $H^s(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open set. The set of smooth, compactly supported functions in Ω is $C^\infty_{\mathrm{c}}(\Omega)$. We set $\mathring{W}^{k,p}(\Omega)$ to be the closure of the $C^\infty_{\mathrm{c}}(\Omega)$ in $W^{k,p}(\Omega)$, and $\mathring{H}^s(\Omega)$ is defined similarly.

Unless otherwise noted, we always take n to be an integer with $n \geq 3$. The ball of radius r about x in $\mathbb{R}^n$ is $B_r(x)$ or simply B_r when $x = 0$; E_r is the region exterior to $\overline{B}_r$; A_r is $B_r \cap E_{\frac{r}{2}}$. If $F : X \to Y$ is a continuous linear map of Banach spaces, we set $||F||_X$ to be the operator norm of F, leaving the target space to be inferred. We define $f^{(+)}(x) = \max(f(x), 0)$. If $s \in \mathbb{R}$, we set $[s]$ to be the largest integer k such that $k \leq s$.

Let $\mathcal{M}$ be a Lorentzian manifold with metric $\mathbf{g}$ and connection $\nabla^{\mathbf{g}}$. If M is a spacelike hypersurface of $\mathcal{M}$ with timelike unit normal N, we define the extrinsic curvature K of M in $\mathcal{M}$ by $K(X, Y) = \left\langle \nabla^{\mathbf{g}}_X Y, N \right\rangle_{\mathbf{g}}$ for vector fields tangent to M. This definition agrees with that used in [YB80] and [Da02], but differs in sign from that used in [Wa84] and [Da03].

Let M be a Riemannian manifold with metric g and connection ∇^g. If Σ is a spacelike hypersurface of M with unit normal ν and induced metric g', the extrinsic curvature k of Σ in M is similarly defined by $k(X,Y) = \left\langle \nabla^g_X Y, \nu \right\rangle_g$. The mean curvature H of Σ computed with respect to ν is $\operatorname{tr}_{g'} k$. This convention for the mean curvature agrees with that typically used in general relativity, but differs by a multiplicative constant from the convention used in, e.g., [Le97]. We set the Laplacian on (M,g) to be $\Delta_g = \operatorname{div}_g \operatorname{grad}_g$, so Δ_g has negative eigenvalues.

We use the notation $A \lesssim B$ to mean $A < cB$ for a certain positive constant c. The constant is independent of the functions and parameters appearing in A and B that are not assumed to have a fixed value. For example, when considering a sequence $\{f_i\}_{i=1}^{\infty}$ of functions on a domain Ω, the expression $||f_i||_{L^1(\Omega)} \lesssim 1$ means the sequence is uniformly bounded in $L^1(\Omega)$ (with a bound that might depend on Ω).

Chapter 2

THE APPARENT HORIZON BOUNDARY CONDITION

2.1 Black Holes and Scri

Heuristically, a black hole is a region of space that cannot send signals to infinity. One way to make this notion precise is the machinery of conformal completions. A Lorentzian manifold with boundary $(\mathcal{M}', \mathbf{g}')$ is a conformal completion of $(\mathcal{M}, \mathbf{g})$ if there exists an embedding $\iota : \mathcal{M} \mapsto \mathcal{M}'$ such that

1. $\iota(\mathcal{M}) = \operatorname{int} \mathcal{M}'$.

2. There exists a differentiable function Ω smooth on $\operatorname{int} \mathcal{M}'$, such that $\mathbf{g}' = \Omega^2 \iota_* \mathbf{g}$ in $\operatorname{int} \mathcal{M}'$.

3. Ω vanishes on $\mathscr{I} = \partial \mathcal{M}'$ and $d\Omega$ is nowhere vanishing on $\mathscr{I}$.

One typically also makes additional hypotheses concerning the smoothness of $\mathscr{I}$ and on the differentiability of Ω at $\mathscr{I}$, but the correct choice of these is still an area of research and is not important to the subsequent exposition. Since Ω vanishes at the boundary, we think of the boundary $\mathscr{I}$ (called *scri*) as the boundary at infinity. Note that if $\mathcal{M}'$ is a conformal completion of $\mathcal{M}$ and if $p \in \mathscr{I}$, then $\mathcal{M}' - \{p\}$ is also a conformal completion of $\mathcal{M}$. Hence any useful definition of conformal completion should also include a completeness hypothesis, which we will return to below.

We first recall the standard conformal completion of Minkowski space $\mathbb{M}^4$, i.e. $\mathbb{R}^4$ equipped with the metric $\operatorname{diag}(-1, 1, 1, 1)$. Consider the cylindrical manifold $E = \mathbb{R} \times S^3$ with the metric $\mathbf{g}' = -dT^2 + dS_3^2$, where dS_3^2 is the round metric on S^3 and T is the coordinate on the $\mathbb{R}$ factor. We then embed $\mathbb{M}^4$ into E as follows. Let t and r denote the usual time and spacial radius coordinates of Minkowski space, and let R denote the distance in S^3 from the north pole. We then set

$$T = \arctan(t + r) + \arctan(t - r)$$

$$R = \arctan(t + r) - \arctan(t - r)$$

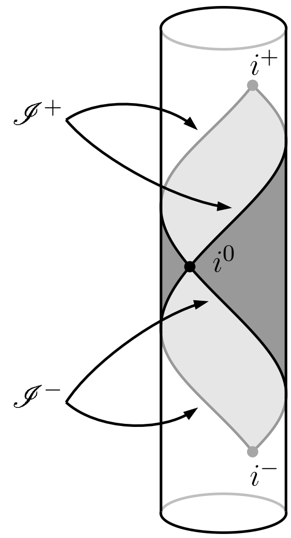

Figure 2.1: The Conformal Completion of $\mathbb{M}^4$ as a subset of $S^3 \times \mathbb{R}$

and identify surfaces of constant r in $\mathbb{M}^4$ with surfaces of constant R in S^3 in the natural way to obtain an embedding of $\mathbb{M}^4$ into the region of E given by

$$-\pi < T + R < \pi$$
$$-\pi < T - R < \pi$$

Letting ι denote this embedding, then

$$\iota^* \mathbf{g}' = \Omega^2 \overline{\mathbf{g}}$$

where

$$\Omega^2 = \frac{4}{(1 + (t + r)^2)(1 + (t - r)^2)}.$$

We take $\mathcal{M}'$ to be the image of $\mathbb{M}$ under this embedding together with the null surfaces $\mathscr{I}^+$ and $\mathscr{I}^-$ indicated in Figure 2.1 (the points i^0, i^+, and i^- are not part of $\mathcal{M}'$; the boundary is not regular at $i^\pm$ and $\mathcal{M}' \cup i^0$ is not a manifold with boundary).

To extend these notions to more general manifolds, we need some concepts from causality theory. We say a point x chronologically precedes y if there exists a piecewise future directed timelike curve from x to y, in which case we write $x \ll y$. Similarly, x causally precedes y if there exists a piecewise future directed causal curve from x to y, and we write $x \prec y$. Note that degenerate curves are never timelike, but are always causal. Hence $x \prec x$ always holds, but it is not true in general that $x \ll x$.

The chronological future of a point x is the set $I^+(x) = \{y : x \ll y\}$, and the chronological past of x is $I^-(x) = \{y : y \ll x\}$. The causal future $J^+(x)$ and past $J^-(x)$ are defined similarly replacing $\ll$ with $\prec$. These definitions extend in the obvious way to sets. For example, $I^+(S) = \cup_{x \in S} I^+(x)$. It follows trivially from the definitions that $I^+(S) \subset J^+(S)$. Moreover by working in geodesic normal coordinates, it is easy to see that $I^+(S)$ is always open. It is not true in general that $J^+(S)$ is closed, even when S is compact. For example, if $x \in \partial(J^-(0))$ in $\mathbb{M}^n$, then $J^+(x)$ is not closed in $\mathbb{M}^n - 0$. So $\overline{I}^+(S) \supset J^+(S) \supset I^+(S)$ with each containment strict in general. But in the case of a globally hyperbolic spacetime $J^+(K) = \overline{I}^+(K)$ for all compact sets K, and in particular $J^+(K)$ is closed.

For a general conformal completion, we define $\mathscr{I}^+ = \{x \in \mathscr{I} : I^- x \cap \mathcal{M} \neq \emptyset\}$, with an analogous definition for $\mathscr{I}^-$. Intuitively $\mathscr{I}^+$ is the portion of the boundary that can be reached by future oriented timelike curves starting in $\mathcal{M}$. These definitions agree with the sets indicated in Figure 2.1 for the conformal completion of $\mathbb{M}^4$.

Given a Lorentzian manifold $\mathcal{M}$ possessing a conformal completion $\mathcal{M}'$, we define the black hole region of $\mathcal{M}$ to be $\mathcal{B} = \mathcal{M} - J^-(\mathscr{I}^+)$, so that the black hole region is the part of $\mathcal{M}$ that cannot be seen from $\mathscr{I}^+$. Obviously, the black hole region depends on the choice of conformal completion. To compensate for this, we say $\mathcal{M}$ is asymptotically Minkowskian at future null infinity if it has a conformal completion $\mathcal{M}'$ and moreover

4. $\operatorname{Hess}\Omega = 0$ on $\mathscr{I}^+$, where the Hessian is computed with respect the conformal metric on $\mathcal{M}'$.

5. $\mathscr{I}^+$ is foliated by complete null geodesics.

6. $\mathscr{I}^+$ is homeomorphic to $S^2 \times \mathbb{R}$, with the homeomorphism taking each null generator to a copy of $\mathbb{R}$.

These technical conditions, due to [GH78], ensure that $\mathscr{I}^+$ "looks like" the $\mathscr{I}^+$ for Minkowski space and is in this sense maximal. If the previous hypotheses also apply to $\mathscr{I}^-$, we say that $\mathcal{M}$ is asymptotically Minkowskian.

Asymptotically Euclidean initial data (M, g, K) can be defined in a similar way using conformal completions modeled on the conformal completion of Euclidean space into the sphere. We choose, however, to use an equivalent definition of asymptotically Euclidean data in terms of preferred charts near infinity and decay properties of g and K in these charts. Chapter 3 contains a precise definition of asymptotically Euclidean initial data.

Given a spacelike hypersurface M of $\mathcal{M}$, we count the number of black holes in M by counting the number of components of $\mathcal{B} \cap M$. Hence the number of black holes in a spacelike hypersurface (if any) is not a local property of the hypersurface, but a global property of the spacetime. This presents a significant challenge for constructing Cauchy data containing black holes. It would be nice to have a way to detect Cauchy data that will form a black hole (without first evolving the data), and this leads us to a discussion of trapped surfaces.

2.2 Trapped Surfaces and Apparent Horizons

Suppose Σ is a compact two-dimensional spacelike submanifold of a four-dimensional Lorentzian manifold $\mathcal{M}$. Then at each point $p \in \Sigma$ there exist a pair of future pointing null vectors N_+ and N_- normal to Σ such that $T_p\mathcal{M} = T_p\Sigma \oplus \operatorname{span} N_+ \oplus \operatorname{span} N_-$. Since $\mathbf{g}(N_\pm, N_\pm) = 0$, we cannot use unit length normalization to select a distinguished choice of future pointing vector in $\operatorname{span} N_\pm$. The choice of vectors N_+ and N_- is unique, however, up to scaling and transposition. If the normal bundle of Σ is orientable, then N_+ and N_- can be extended to a pair of smooth future pointing null normal vector fields on Σ. Since $\mathcal{M}$ is time-orientable, the normal bundle of Σ is orientable if and only if there exists a globally defined spacelike unit normal vector ν to Σ. In particular, if Σ is the boundary of a spacelike hypersurface M, then the normal bundle of Σ is orientable. For simplicity we will assume that both the normal and tangent bundles of Σ are orientable.

We can construct a null hypersurface $\mathcal{N}_+$ containing Σ and ruled by null geodesics such that N_+ is tangent to these geodesics. The vector field N_+ on Σ extends naturally to a vector field, also called N_+, tangent to $\mathcal{N}_+$ and satisfying $\nabla_{N_+} N_+ = 0$ (here ∇ is the connection on the ambient Lorentzian manifold and N_+ is suitably extended off of $\mathcal{N}$; such extensions will be made as needed without comment henceforth). On Σ we have the null second fundamental form χ_+ given by $\chi_+(X,Y) = -\langle\nabla_X N_+, Y\rangle$. Formally this expression resembles the second fundamental form of Riemannian geometry. However, since $\langle N_+, N_+\rangle = 0$, $\chi_+(X,Y)$ detects, in some sense, the part of $\nabla_X Y$ in $\operatorname{span} N_-$. The quantity $\theta_+ = -\operatorname{tr}\chi_+$ is known as the convergence or expansion of Σ with respect to N_+ and plays a similar role to mean curvature in Riemannian geometry. Using the flow of N_+ to define a family of spacelike surfaces Σ_t with $\Sigma_0 = \Sigma$ we see that θ_+ is well defined on $\mathcal{N}$, not just on Σ.

Now $\mathcal{N}$ possesses a unique globally defined two-form dA satisfying $dA(E_1, E_2) = 1$ for any pair of oriented orthonormal spacelike basis vectors to $\mathcal{N}$. An easy computation shows $\mathcal{L}_{N_+} dA = \theta_+ dA$. So θ_+ describes the change in area of Σ as it evolves under the flow of N_+. Since N_+ is defined only up to scale, it is interesting to note how θ_+ depends on N_+. The scale of N_+ defines a choice of affine parameter for the null geodesics that rule $\mathcal{N}$. If instead of N_+ we work with λN_+, where λ is a positive function on Σ, we obtain

$$\begin{aligned}\chi_{N_+}(X,Y) &= \langle\nabla_X \lambda N_+, Y\rangle\\ &= X\lambda\,\langle N_+, Y\rangle + \lambda\,\langle\nabla_X N_+, Y\rangle\\ &= \lambda\,\chi_{N_+}(X,Y).\end{aligned}$$

In particular, $\theta_{\lambda N_+} = \lambda\theta_{N_+}$ and hence the sign of θ_+ is geometrically significant. We say a surface Σ is trapped if both $\theta_\pm < 0$ and marginally trapped if both $\theta_\pm \leq 0$ on all of Σ. A trapped surface is (instantaneously) shrinking as it evolves under the flow of any family of orthogonal future pointing null geodesics.

To see how θ_+ evolves under the flow of N_+, we compute $\nabla_{N_+}\operatorname{tr}\chi_+ = \operatorname{tr}\nabla_{N_+}\chi_+$. Doing so, we arrive at

$$\begin{aligned}\nabla_{N_+}\theta_+ &= -|\chi_+|^2 - \operatorname{Ric}(N_+, N_+)\\ &= -\frac{1}{2}\theta^2 - |\sigma_+|^2 - \operatorname{Ric}(N_+, N_+)\end{aligned}\tag{2.1}$$

where σ_+ is the trace free part of χ_+ and Ric is the Ricci tensor of the ambient Lorentzian metric. Equation (2.1) is known as the Raychaudhuri equation. If $\operatorname{Ric}(N,N) \geq 0$ for all null vectors N (this is known as the null energy condition) then we can compare (2.1) with the ODE $x' = -x^2$. It follows from Gronwall's inequality that if $\theta_+ = \theta_0 < 0$ at affine parameter $t = 0$, then θ_+ tends to $-\infty$ somewhere within $t \in (0, -2/\theta_0]$.

Points where $\theta_+ = -\infty$ are important because of a connection with conjugate points. The family of null geodesics generating $\mathcal{N}_+$ is associated with a set of Jacobi fields arising from deviations through this family. A point p on an orthogonal null geodesic γ starting at Σ is said to be conjugate to Σ if there exists a nontrivial Jacobi field X arising from such a deviation that vanishes at p. It turns out that a necessary and sufficient condition for p to be conjugate to Σ is $\theta_+(p) = -\infty$. This implies that every null geodesic orthogonal to a trapped surface Σ has a conjugate point within a finite affine parameter from Σ (so long as the geodesic can be extended that far), and since Σ is compact we have a uniform bound on this parameter. One can also show that if p is conjugate to Σ, then the geodesic γ connecting p and Σ is homotopic to a *timelike* curve connecting p to Σ and hence $p \in I^+(\Sigma)$.

Trapped surfaces play an important role in the theory of gravitational collapse and the appearance of singularities. The following theorem of Penrose is typical. We sketch to proof to see the relationship between trapped surfaces, the appearance of conjugate points, and the appearance of singularities.

Theorem 2.1 [Pe65] *Let $\mathcal{M}$ be a spacetime satisfying the null energy condition. If $\mathcal{M}$ has a non-compact Cauchy surface M containing a trapped surface Σ, then there exists an inextendible null geodesic starting at Σ that terminates within finite affine parameter.*

Idea of Proof: Since $\mathcal{M}$ is globally hyperbolic, $J^+(\Sigma) = \overline{I^+(\Sigma)}$ and $\partial I^+(\Sigma)$ is ruled by null geodesics orthogonal to and terminating at Σ. Since Σ is trapped, each such geodesic has a conjugate point p within finite affine parameter hence $p \in I^+(\Sigma)$ and therefore $p \notin \partial I^+(\Sigma)$. It follows that $\partial I^+(\Sigma)$ is homeomorphic to a closed, bounded subset of $\Sigma \times \mathbb{R}$ and is hence compact. On the other hand, since M is a Cauchy surface we can establish a homeomorphism between $\partial I^+(\Sigma)$ and a subset S of M. Since $\partial I^+(\Sigma)$ is compact, so is S and in particular S is closed. Since $\partial I^+(\Sigma)$ is a manifold, S is locally Euclidean and hence open. Since M is connected, $S = M$, and since M is not compact we have a contradiction. $\square$

In particular, Theorem 2.1 shows that if an asymptotically Euclidean initial data set contains a trapped surface, then the resulting maximal globally hyperbolic Cauchy development is not null complete. The theorem does not indicate the cause of the null incompleteness. One possibility that a singularity forms. Another is that the maximal development $(\mathcal{M}, \mathbf{g})$ can be embedded in a larger (not globally hyperbolic) spacetime $(\mathcal{M}', \mathbf{g}')$ In this second case, the boundary of $\mathcal{M}$ in $\mathcal{M}'$ is called a Cauchy horizon. There exists a stronger singularity theorem due to Hawking and Penrose [HP70] that does away with the Cauchy horizon possibility at the expense of adding more hypotheses. Regardless, a trapped surface is associated with pathological behaviour of the resulting Cauchy development. Now, a marginally trapped surface cannot be used in the previous proof since we have no guarantee that a marginally trapped surface generates null geodesics having conjugate points. On the other hand, we see from (2.1) that the condition $\theta_\pm = 0$ is not stable. For example, if $\chi_\pm \neq 0$ everywhere on Σ, then again we can conclude the existence of conjugate points. Hence marginally trapped surfaces are also of interest.

Another kind of surface, related to trapped surfaces, is an apparent horizon. Suppose Σ is contained in an asymptotically Euclidean Cauchy surface M and is the boundary of a region M_∞ of M containing spacelike infinity. We can then distinguish N_+ and N_- by the condition $\langle N_+, \nu \rangle < 0$, where ν is the outward pointing normal vector of M_∞. We say that such a Σ is outer marginally trapped if $\theta_+ \leq 0$ and is an apparent horizon if θ_+ vanishes identically. Apparent horizons are particularly interesting, since the boundary of a maximal foliation by trapped surfaces can be shown to be an apparent horizon, assuming the boundary is sufficiently smooth. In this sense, apparent horizons play a role for trapped surfaces that the event horizon does for the true boundary of a black hole.

One can show that trapped, marginally trapped and outer marginally trapped surfaces all signal the development of a black hole. This is important, since the expansion of a surface Σ in M can be computed directly using the initial data (M, g, K). Of course, the claim that a black hole appears is contingent on the resulting Cauchy evolution having an appropriate scri. That this is true generically is known as the weak cosmic censorship which can be roughly formulated in the vacuum setting as follows.

Weak Cosmic Censorship Conjecture Let (g, K, M) be asymptotically Euclidean vacuum initial data satisfying appropriate smoothness and decay hypotheses. Then *generically* the maximal Cauchy evolution $(\mathcal{M}, \mathbf{g})$ of this data is asymptotically Minkowskian at future null infinity and $\mathcal{M} \cup \mathscr{I}^+$ is globally hyperbolic.

The heuristic idea behind the conjecture is straightforward. If a complete $\mathscr{I}^+$ forms, observers sufficiently near infinity will never be affected by a pathology that develops in the spacetime. Moreover, since $\mathscr{I}^+$ is contained in the domain of dependence of M, no pathologies are visible at infinity. So if any exist, they must be contained in a black hole. The generic caveat is present in the conjecture since it is know that certain spherically symmetric initial data for gravity coupled with a Klein-Gordon matter field do form naked singularities [Ch94], but that these singularities do not persist under perturbations. In the vacuum setting, no known counterexamples to weak cosmic censorship exist. We have the following theorem relating trapped surfaces and black holes

Theorem 2.2 [HE73][Wa84] *Let (g, K, M) be initial data containing a trapped, marginally trapped, or marginally outer trapped surface Σ. If the maximal Cauchy development $(\mathcal{M}, \mathbf{g})$ of the data is weakly censored, then $I^-(\mathscr{I}^+) \cap \Sigma = \emptyset$ and hence Σ is contained in the black hole region of $(\mathcal{M}, \mathbf{g})$.*

Although this result is widely accepted in the relativity community, the proofs in the cited texts are not correct (except for [Wa84] in the context of trapped surfaces and with mildly stronger hypotheses than those stated here). One can, in fact, give a correct proof for trapped surfaces. The case of marginally (outer) trapped surfaces is more delicate but also can be addressed [CG03].

2.3 Initial Data Containing Black Holes

Theorem 2.2 motivates finding finding solutions (M, g, K) of the constraint equations (1.4) and (1.5) containing trapped surfaces or apparent horizons. Let (M, g, K) be an asymptotically Euclidean data set, and let Σ be an orientable compact hypersurface with unit normal ν. If M is a Cauchy surface for a spacetime $\mathcal{M}$ with future pointing timelike unit normal n, then the vectors $N_\pm = \pm\nu + n$ are null, future pointing, and orthogonal to Σ. The convergences $\theta_\pm$ computed with respect to $N_\pm$ are

$$\theta_\pm = -\operatorname{tr} K + K(\nu, \nu) \mp H,$$

where H is the mean curvature of Σ in M computed with respect to ν. For time symmetric initial data (that is, data with $K = 0$), then the equation $\theta_+ = 0$ reduces to $H = 0$. The Hamiltonian constraint (1.4) reduces to $R = 0$ and the momentum constraint (1.5) is satisfied automatically. So a

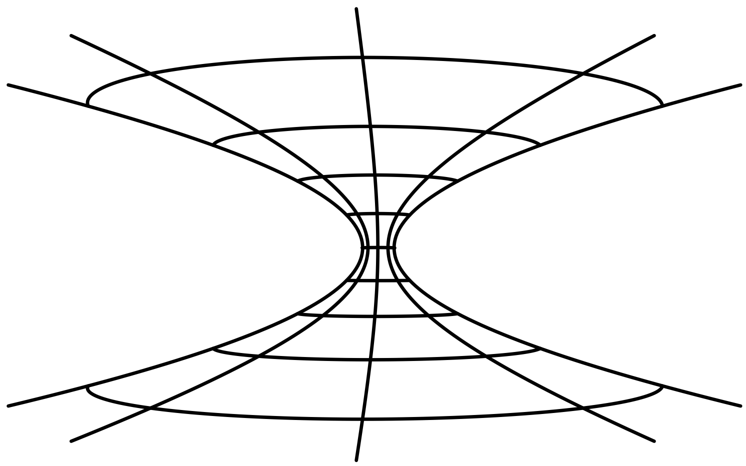

Figure 2.2: Time Symmetric Slice of Schwarzschild

time symmetric solutions of the constraints with an apparent horizon is just a scalar flat Riemannian manifold with a minimal surface. Consider the manifold $\mathbb{R} \times S^2$ with the metric $(1+s^2)ds^2 + \left(\frac{s^2+1}{2}\right)^2 dS^2$, where s is the coordinate along $\mathbb{R}$ and dS^2 is the round metric on the sphere. One readily shows this manifold is scalar flat and that the level set $s = 0$ is a minimal surface. The maximal development of this data is a member the one parameter family of Schwarzschild solutions of the Einstein equations, and is a prototypical black hole solution.

Note from Figure 2.2 that the Schwarzschild initial data has two asymptotically Euclidean ends connected by a neck. For the purposes of computing the black hole region $\mathcal{B}$ of the resulting Cauchy development, we select a distinguished end and work with the scri of that end.

One can imagine similar data formed by connecting two asymptotically Euclidean regions together with several necks, or connecting several asymptotically Euclidean regions to a given distinguished one. Schemes such as those in in [Mi63], [BL63], [YB80] (see also [Ck00]) create families of initial data containing apparent horizons or trapped surfaces inside necks. A very flexible approach for generating necks comes from a gluing construction [IMP02]. One can, for example, start with two asymptotically Euclidean solutions of the constraints (M_i, g_i, K_i), $i = 1, 2$, and generate a third solution $(M_1 \# M_2, g, K)$ on the connected sum. The new solution will contain a neck and will closely approximate the original solutions away from the surgery location. In certain cases, one can prove rigorously that necks introduced this way will evolve into distinct black holes [CM03].

The previous methods for generating black hole initial data create apparent horizons indirectly by topological means. A direct approach for creating apparent horizons, introduced by Thornberg [Th87], is to work with a manifold with boundary and prescribing that the boundary be an apparent horizon. Thornburg numerically investigated generating such initial data, and variations of the apparent horizon condition have subsequently been proposed for numerical study, e.g. [Ck02] [Ea98]. However, as indicated by Dain [Da02], there has not been a rigorous mathematical investigation of the apparent horizon boundary condition.

Let (M, g, K) be an asymptotically Euclidean data set on a manifold with compact boundary, and let ν denote the exterior unit normal to ∂M. The convergences $\theta_\pm$ computed at the boundary are

$$\theta_\pm = -\operatorname{tr} K + K(\nu,\nu) \mp H,$$

where H is the mean curvature of ∂M in M computed with respect to $-\nu$. The convergence θ_+ corresponds to the outgoing (to infinity) null direction and hence the boundary is an apparent horizon if $\theta_+ = 0$. So our goal is to find initial data satisfying

$$\begin{aligned} R - |K|^2 + \operatorname{tr} K^2 &= 0 \\ \operatorname{div} K - d\ \operatorname{tr} K &= 0 \\ -\operatorname{tr} K + K(\nu,\nu) - H &= 0 \quad \text{on } \partial M. \end{aligned} \tag{2.2}$$

2.4 Construction Via the Conformal Method

Following the strategy for manifolds without boundary, we start with an asymptotically Euclidean manifold (M^3, g) with boundary and a transverse traceless tensor σ. We then seek to find a conformal factor ϕ such that $\tilde{g} = \phi^4 g$ and $\tilde{K} = \phi^{-2}\sigma$ solves the constraint equations, and we want ∂M to satisfy $\tilde{\theta}_+ = 0$. The convergences $\tilde{\theta}_\pm$ can be written in terms of the conformal data, and in particular the condition $\tilde{\theta}_+ = 0$ becomes

$$4\phi^{-3}\partial_\nu \phi + H\phi^{-2} - \phi^{-6}\sigma(\nu,\nu) = 0$$

where ν is the exterior unit normal to ∂M and H is computed with respect to $-\nu$. So we want to find conditions on (M, g, σ) under which the boundary value problem

$$\begin{aligned} -8\Delta\,\phi + R\phi - |\sigma|^2\,\phi^{-7} &= 0 \\ 4\partial_\nu\phi + H\phi - \phi^{-3}\sigma(\nu,\nu) &= 0 \quad \text{on } \partial M \end{aligned} \tag{2.3}$$

is solvable. Given the analysis of the Lichnerowicz equation in the case $\partial M = \emptyset$, it seems reasonable that there will be a restriction on the conformal class $[g]$. We express this restriction in terms of a conformal invariant for asymptotically Euclidean manifolds with boundary that generalizes one introduced by Escobar [Es92] for compact manifolds. On 3-manifolds, the invariant is

$$\lambda_g = \inf_{f\in C_c^\infty(M), f\not\equiv 0} \frac{\int_M 8\,|\nabla f|^2 + Rf^2\,dV + \int_{\partial M} 2Hf^2\,dA}{||f||_{L^6}^2}.$$

In Chapter 6 we prove that there exists a solution of (2.3) provided

1. (M, g) satisfies $\lambda_g > 0$, $R = 0$ and $H < 0$,

2. σ satisfies $H \le \sigma(\nu,\nu) \le 0$,

and that there exists a large class of conformal data (M, g, σ) satisfying conditions 1–2. For manifolds without boundary, $\lambda_g > 0$ is a necessary condition. Although it is not clear if this condition is also necessary to solve the boundary value problem (2.3), our construction requires it because it ensures

$$\mathcal{P} = (-8\Delta\,+R, 4\partial_\nu + H\,|_{\partial M})$$

is an isomorphism acting on certain weighted Sobolev spaces. We also show that if $\lambda_g > 0$, then we can always conformally change to an asymptotically Euclidean manifold satisfying $R = 0$ and $H < 0$. So in some sense the requirements $R = 0$ and $H < 0$ are superfluous. We make these requirements explicit, however, since condition 2 is not conformally invariant; the inequality in condition 2 must hold with respect to a conformal representative having $R = 0$ and $H < 0$.

To motivate condition 2, we first consider the sign condition $\sigma(\nu,\nu) \le 0$. Since $\tilde{K}(\tilde{\nu},\tilde{\nu}) = \phi^{-6}\sigma(\nu,\nu)$, it follows that the sign of $\sigma(\nu,\nu)$ determines the sign of $\tilde{K}(\tilde{\nu},\tilde{\nu})$. Now if $\tilde{\theta}_+ = 0$, it follows that

$$\tilde{K}(\tilde{\nu},\tilde{\nu}) = \tilde{H}.$$

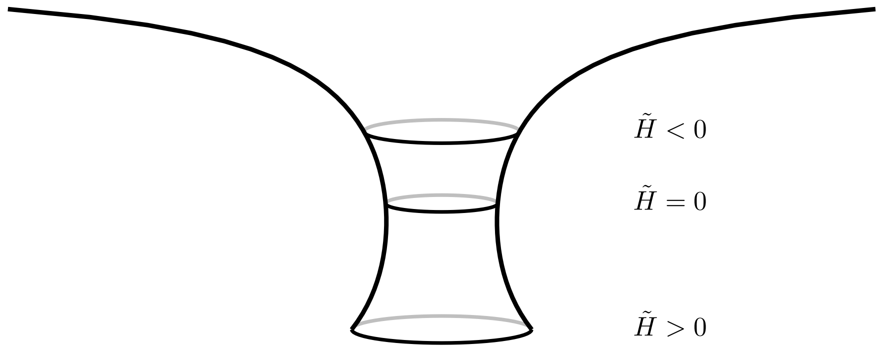

Figure 2.3: Boundary Mean Curvatures of an Asymptotically Euclidean Manifold

Thus the sign of $\sigma(\nu,\nu)$ also determines the sign of $\tilde{H}$. Finally, the sign of $\tilde{H}$ determines a relationship between $\tilde{\theta}_+$ and $\tilde{\theta}_-$. Since

$$\tilde{\theta}_+ = \tilde{K}(\tilde{\nu},\tilde{\nu}) - \tilde{H}$$
$$\tilde{\theta}_- = \tilde{K}(\tilde{\nu},\tilde{\nu}) + \tilde{H},$$

we conclude that $\tilde{H} \leq 0$ implies $\tilde{\theta}_+ \geq \tilde{\theta}_-$ whereas $\tilde{H} \geq 0$ implies $\tilde{\theta}_+ \leq \tilde{\theta}_-$. Hence we require $\sigma(\nu,\nu) \leq 0$ to ensure $\tilde{\theta}_- \leq \tilde{\theta}_+ = 0$ and therefore the boundary of M is not only an apparent horizon, but also a marginally trapped surface. In fact, an earlier version of [Ma03] worked with the condition $\sigma(\nu,\nu) \geq 0$. Although this allowed for the construction of apparent horizons, it was observed in [Da03] that these surfaces are of limited physical interest because they are not marginally trapped surfaces. For comparison, we consider the approach of [Da03]. Rather than work with $\tilde{\theta}_+$, Dain prescribes $\tilde{\theta}_- \leq 0$ and under suitable conditions constructs solutions satisfying $\tilde{\theta}_+ \leq \tilde{\theta}_- \leq 0$. These are trapped surfaces, but the relationship $\tilde{\theta}_+ \leq \tilde{\theta}_-$ shows that for these solutions $\tilde{H} \geq 0$. Hence the method of [Da03] cannot construct an apparent horizon (except in the extremal case $\tilde{\theta}_+ = \tilde{\theta}_- = \tilde{H} = 0$). To create an apparent horizon that is also a marginally trapped surface, we must have $\tilde{H} \leq 0$. Figure 2.3 shows boundaries with mean curvatures of different signs and indicates the difference between the conditions $\tilde{H} \geq 0$ and $\tilde{H} \leq 0$.

We now analyze to the condition $H \leq \sigma(\nu,\nu)$. Since $\sigma(\nu,\nu) \leq 0$, we have the necessary consequence $H \leq 0$. From an analysis point of view, we would rather have $H \geq 0$. For example, if $R \geq 0$ and $H \geq 0$, then there is a maximum principle associated with $\mathcal{P}$. This would be a useful tool to show conformal factors we construct are positive. This last fact is part of the motivation for the condition $\sigma(\nu,\nu) \geq 0$ in the prior version of [Ma03] and also for the choice in [Da03] to work with $\tilde{\theta}_-$ rather than $\tilde{\theta}_+$. The inequality $H \leq \sigma(\nu,\nu)$ is used to compensate for the loss of the maximum principle. To understand the meaning of this condition, we note that $\sigma(\nu,\nu) - H$ is equivalent to $\theta_+ > 0$ for the conformal data. Hence we can start with conformal data with $\theta_+ > 0$ and $\theta_- < \theta_+$ and transform to initial data with $\tilde{\theta}_+ = 0$ and $\tilde{\theta}_- < \tilde{\theta}_+$.

Chapter 3

WEIGHTED SOBOLEV SPACES

3.1 Motivation

On a bounded open set Ω, the Laplacian is an isomorphism from $\mathring{W}^{2,2}(\Omega)$ to $L^2(\Omega)$. By contrast, the Laplacian does not have good mapping properties from $\mathring{W}^{2,2}(\mathbb{R}^n) = W^{2,2}(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. It is not a Fredholm operator and in particular it does not have closed range. To see this, we first note that the Laplacian has trivial kernel in $W^{2,2}(\mathbb{R}^n)$. For if $\Delta\, u = 0$, then

$$0 = \int_{\mathbb{R}^n} -u\Delta\, u\; dV = \int_{\mathbb{R}^n} |\nabla u|^2 \;\, dV$$

and hence $u = 0$. If the Laplacian had closed range $X \subset L^2(\mathbb{R}^n)$, then X would be a Banach space with the L^2 norm. We would then have an isomorphism $\Delta\ : W^{2,2}(\mathbb{R}^n) \to X$ and the resulting inequality

$$||u||_{W^{2,2}(\mathbb{R}^n)} \lesssim ||\Delta\, u||_{L^2(\mathbb{R}^n)}$$

for all $u \in W^{2,2}(\mathbb{R}^n)$. A scaling argument shows that this inequality cannot hold. For $r > 0$ let $\mathcal{S}_r$ be the rescaling operator $(\mathcal{S}_r u)(x) = u(rx)$. The rescaling operator takes a function supported on B_r to a function supported on B_1. For $u \in W^{2,2}(\mathbb{R}^n)$ we have $\Delta\, \mathcal{S}_r u = r^2 \mathcal{S}_r \Delta\, u$ and $||\mathcal{S}_r u||_{L^2(\mathbb{R}^n)} = r^{-\frac{n}{2}} ||u||_{L^2(\mathbb{R}^n)}$. So

$$\begin{aligned} ||u||_{L^2(\mathbb{R}^n)} = r^{-\frac{n}{2}} ||\mathcal{S}_{\frac{1}{r}} u||_{L^2(\mathbb{R}^n)} &\leq r^{-\frac{n}{2}} ||\mathcal{S}_{\frac{1}{r}} u||_{W^{2,2}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{n}{2}} ||\Delta\, \mathcal{S}_{\frac{1}{r}} u||_{L^2(\mathbb{R}^n)} = r^{-2} ||\Delta\, u||_{L^2(\mathbb{R}^n)}. \end{aligned}$$

This is obviously false for large r. We require function spaces with better scaling properties to study the Laplacian on $\mathbb{R}^n$.

In [Mc79], building on earlier work by Nirenberg and Walker in [NW73], McOwen showed that the Laplacian on $\mathbb{R}^n$ does have good mapping properties on weighted Sobolev spaces. Letting

$\langle x \rangle = (1+|x|^2)^{1/2}$ we define for $k \in \mathbb{Z}_{\geq 0}$, $1 \leq p < \infty$, and $\delta \in \mathbb{R}$

$$W^{k,p}_\delta(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^* : ||u||_{W^{k,p}_\delta(\mathbb{R}^n)} = \sum_{|\beta| \leq k} || \langle x \rangle^{-\delta - \frac{n}{p} + |\beta|} \, \partial_\beta u ||_{L^p(\mathbb{R}^n)} < \infty \right\}$$
$$L^p_\delta(\mathbb{R}^n) = W^{0,p}_\delta(\mathbb{R}^n).$$

Heuristically, a function in $W^{k,p}_\delta$ has growth at infinity no faster than $O\left(|x|^\delta\right)$. For example, $\langle x \rangle^{\delta'} \in W^{k,p}_\delta$ for every $\delta' < \delta$, but $\langle x \rangle^\delta \notin W^{k,p}_\delta$. More importantly, differentiation takes $W^{k,p}_\delta$ to $W^{k-1,p}_{\delta-1}$, so the derivative of a function in $W^{k,p}_\delta$ looses an order of growth at infinity. As we will see, this second property is central to the good scaling properties of these spaces.

Let A_r be the annulus $B_r \setminus \overline{B_{\frac{r}{2}}}$. Then on A_r we have $m_\alpha r^\alpha \leq w(x)^\alpha \leq M_\alpha r^\alpha$ where m_α and M_α are independent of $r > 1$. So

$$r^{-p\delta - n + p|\beta|} ||\partial_\beta u||^p_{L^p(A_r)} \lesssim ||w(x)^{-\delta - \frac{n}{p} + |\beta|} \, \partial_\beta u||^p_{L^p(A_r)} \lesssim r^{-p\delta - n + p|\beta|} ||\partial_\beta u||^p_{L^p(A_r)},$$

where the implicit constants are independent of $r \geq 1$ and u. Since

$$\int_{A_r} \left| \partial^\beta u \right|^p \, dV = r^{n - p|\beta|} \int_{A_1} \left| \partial^\beta \mathcal{S}_r u \right|^p \, dV$$

we find

$$r^{-p\delta} ||\partial_\beta \mathcal{S}_r u||^p_{L^p(A_1)} \lesssim ||w(x)^{-\delta - \frac{n}{p} + |\beta|} \, \partial_\beta u||^p_{L^p(A_r)} \lesssim r^{-p\delta} ||\partial_\beta \mathcal{S}_r u||^p_{L^p(A_1)}.$$

Thus an equivalent norm for the norm on $W^{k,p}_\delta$ is

$$||u||^p_{\tilde{W}^{k,p}_\delta} = ||u||^p_{W^{k,p}(B_1)} + \sum_{j=1}^\infty 2^{-p\delta j} ||\mathcal{S}_{2^j} u||^p_{W^{k,p}(A_1)}. \tag{3.1}$$

This form of the norm, as indicated in [Ba86], makes computations with these spaces more transparent. For example, we have from interior elliptic estimates

$$\begin{aligned} ||\mathcal{S}_{2^j} u||_{W^{k,p}(A_1)} &\lesssim ||\Delta \, \mathcal{S}_{2^j} u||_{W^{k-2,p}(B_2 \setminus B_{\frac{1}{4}})} + ||\mathcal{S}_{2^j} u||_{L^p(B_2 \setminus B_{\frac{1}{4}})} \\ &\lesssim 2^{2j} ||\mathcal{S}_{2^j} \Delta \, u||_{W^{k-2,p}(B_2 \setminus B_{\frac{1}{4}})} + ||\mathcal{S}_{2^j} u||_{L^p(B_2 \setminus B_{\frac{1}{4}})} \end{aligned}$$

Since $B_2 \setminus B_{\frac{1}{4}} \subset \overline{A_{\frac{1}{2}} \cup A_1 \cup A_2}$, an easy computation shows

$$||\mathcal{S}_{2^j} u||^p_{W^{k,p}(A_1)} \lesssim \sum_{i=-1}^{1} 2^{2p(i+j)} ||\mathcal{S}_{2^{i+j}} \Delta \, u||^p_{W^{k-2,p}(A_1)} + \sum_{i=-1}^{1} ||\mathcal{S}_{2^{i+j}} u||^p_{L^p(A_1)}$$

and we conclude

$$||u||_{W^{k,p}_\delta} \lesssim ||\Delta\, u||_{W^{k-2,p}_{\delta-2}} + ||u||_{L^p_\delta}.$$

If the embedding $W^{k,p}_\delta \subset L^p_\delta$ were compact, this inequality would imply $\Delta : W^{k,p}_\delta \to W^{k-2,p}_{\delta-2}$ is semi-Fredholm (i.e. has closed range and finite dimensional kernel). In fact, this embedding sits at the threshold of being compact, and McOwen [Mc79] showed that for all but a few values of δ, $\Delta : W^{k,p}_\delta \to W^{k-2,p}_{\delta-2}$ is indeed Fredholm. We will say δ is exceptional if $\delta \in \mathbb{Z}$ and $\delta \geq 0$ or $\delta \leq 2-n$. We have

Theorem 3.1 [Mc79] *Suppose $\delta \in \mathbb{R}$ is non-exceptional, $1 < p < \infty$ and $k \geq 2$. Then $\Delta : W^{k,p}_\delta(\mathbb{R}^n) \to W^{k-2,p}_{\delta-2}(\mathbb{R}^n)$ is Fredholm and*

$$||u||_{W^{k,p}_\delta} \lesssim ||\Delta\, u||_{W^{k-2,p}_{\delta-2}} + ||u||_{L^p_{\delta'}}$$

for every $\delta' \in \mathbb{R}$. This map is injective when $\delta < 0$ and is surjective when $\delta > 2-n$. In particular, it is an isomorphism when $2-n < \delta < 0$.

The exceptional values $\delta = 0, 1, 2, \cdots$ arise from the harmonic polynomials with these orders of growth. For example, the trouble at $\delta = 0$ comes from the constants which belong to $W^{k,p}_\delta$ for $\delta > 0$ but not for $\delta < 0$. To see that $\Delta : W^{2,p}_0 \to L^p_{-2}$ cannot be Fredholm we first note that if u were harmonic and in $W^{2,p}_0$, then from (3.1) and interior estimates one can show that u is smooth and bounded. But a bounded harmonic function is constant, and the constants do not belong to $W^{2,p}_0$. So Δ has trivial kernel on $W^{2,p}_0$. If it were Fredholm, it would have closed range and we would have the estimate

$$||u||_{W^{2,p}_0} \lesssim ||\Delta\, u||_{W^{0,p}_{-2}}. \tag{3.2}$$

Let u be a function equal to 1 on B_1 and equal to 0 outside B_2. Then $\mathcal{S}_{\frac{1}{r}} u$ is equal to 1 on B_r, and this family of functions can be thought of as an approximating a constant function that nearly belongs to $W^{2,p}_0$. It is easy to see that for $r > 1$

$$\log(r) \lesssim ||\mathcal{S}_{\frac{1}{r}} u||^p_{L^p_0} \leq ||\mathcal{S}_{\frac{1}{r}} u||^p_{W^{2,p}_0}. \tag{3.3}$$

On the other hand, $\Delta\, \mathcal{S}_{\frac{1}{r}} u$ is supported in the annulus A_{2r}. Rescaling we find

$$||\Delta\, \mathcal{S}_{\frac{1}{r}} u||_{L^p_{-2}(\mathbb{R}^n)} \lesssim ||\Delta\, u||_{L^p(A_1)}. \tag{3.4}$$

From (3.2), (3.3) and (3.4) we obtain the impossible estimate $\log(r) \lesssim 1$ for all $r \geq 1$. So Δ cannot be Fredholm for $\delta = 0$.

Since the constants belong to $W^{2,p}_\delta$ for $\delta > 0$, it follows that Δ is not injective on these spaces. A duality argument shows that Δ is not surjective for $\delta < 2-n$. To see this, we consider $\Delta : W^{2,p}_\delta \to L^p_{\delta-2}$. The dual space of $L^p_{\delta-2}$ is $L^{p'}_{2-n-\delta}$ where p' is the usual dual exponent to p. In particular, if $\delta < 2-n$, then $2-n-\delta > 0$ and hence $1 \in L^{p'}_{2-n-\delta}$. So if $f = \Delta\, u$ we have

$$\langle 1, f\rangle = \langle 1, \Delta\, u\rangle = \langle \Delta\, 1, u\rangle = 0$$

where the integration by parts is easily justified by approximating u with smooth compactly supported functions. More succinctly, if $\delta < 2-n$ then the adjoint of $\Delta : W^{2,p}_\delta \to L^p_{\delta-2}$ has nontrivial kernel and hence Δ cannot be onto. The exceptional values $\delta = 2-n, 1-n, -n, \cdots$ are dual to those at $\delta = 0, 1, 2, \cdots$. The most important range of values of δ is $(2-n, 0)$, for in this range Δ is an isomorphism. Our estimates for the scalar and vector Laplacians will all be restricted to the isomorphism range.

Weighted function spaces have been used to study elliptic operators on non-compact manifolds in a number of contexts, e.g. [Ca79b] [CBC81] [LM85] [Ba86]. All of these works use the spaces $W^{k,p}_\delta$ defined above. Our primary tool in investigating low regularity solutions of the constraint equations, however, is a little-used generalization of these spaces more closely related to the spaces H^s where $s \in \mathbb{R}$. Before we examine the properties of these spaces, it is helpful to start with a brief review of the more familiar spaces $W^{k,p}_\delta$.

3.2 Properties of L^p based weighted Sobolev spaces

The following properties can be found in or easily follow from facts in [Ba86]. The principal tool in establishing all of these claims is the alternate norm $\tilde{W}^{k,p}_\delta$ in (3.1).

Lemma 3.2

1. *If $p \geq q$ and $\delta' < \delta$ then $L^p_{\delta'}(\mathbb{R}^n) \subset L^q_\delta(\mathbb{R}^n)$.*

2. *For $k \geq 1$ and $\delta' < \delta$ the inclusion $W^{k,p}_{\delta'}(\mathbb{R}^n) \subset W^{k-1,p}_\delta(\mathbb{R}^n)$ is compact.*

3. *If* $p < n/k$ *then* $W^{k,p}_\delta(\mathbb{R}^n) \subset L^r_\delta(\mathbb{R}^n)$ *for every* r *with* $\frac{1}{p} - \frac{k}{n} \le \frac{1}{r} \le \frac{1}{p}$.

4. *If* $p = n/k$ *then* $W^{k,p}_\delta(\mathbb{R}^n) \subset L^r_\delta(\mathbb{R}^n)$ *for all* $r \ge p$.

5. *If* $1/p < k/n$ *then* $W^{k,p}_\delta(\mathbb{R}^n) \subset L^r_\delta(\mathbb{R}^n)$ *for all* $r \ge p$ *and* $W^{k,p}_\delta(\mathbb{R}^n) \subset C^0_\delta(\mathbb{R}^n)$.

5. *If* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \le 1$, *then pointwise multiplication is a continuous bilinear map* $L^{p_1}_{\delta_1} \times L^{p_2}_{\delta_2} \to L^r_{\delta_1+\delta_2}$.

6. *Pointwise multiplication is a continuous bilinear map* $C^0_{\delta_1} \times L^p_{\delta_2} \to L^p_{\delta_1+\delta_2}$.

Moreover, all the inclusions mentioned above are continuous.

For the most part, intuition about Sobolev spaces on bounded domains transfers over to the weighted setting. There are a couple of points that need care, however. First, we do not have a continuous embedding $L^p_\delta \subset L^q_\delta$ for $p \ne q$ (compare with property 1 of Lemma 3.2). When $p < q$, the failure arises since $L^p \not\subset L^q$, even on compact sets. When $p > q$ the failure stems instead from the fact that l^p is not contained in l^q. In general, the norm on a $W^{k,p}_\delta$ space can control the norm on a $W^{l,q}_\delta$ space without a loss of decay only if $p \le q$. For example, the Sobolev embedding $||u||_{W^{k-1,r}_\delta} \lesssim ||u||_{W^{k,p}_\delta}$ follows this rule of thumb because $r \ge p$.

The second fact that needs care is that the embedding $W^{k,p}_\delta \subset W^{k-1,p}_\delta$ is continuous, but is not compact. To see this, let u be any function with support contained in A_1 and let $u_j = 2^{\delta j}\mathcal{S}_{2^{-j}}u$. Then $||u_j||_{\tilde{W}^{k,p}_\delta} = ||u||_{W^{k,p}(A_1)}$ and $||u_j||_{\tilde{W}^{k-1,p}_\delta} = ||u||_{W^{k-1,p}(A_1)}$. So the sequence $\{u_j\}_{j=1}^\infty$ is bounded in $W^{k,p}_\delta$, and if it had a convergent subsequence in $W^{k-1,p}_\delta$, the limit function would have non-zero norm. But u_k converges to 0 uniformly on compact sets. The embedding $W^{k,p}_{\delta'} \subset W^{k-1,p}_\delta$ is compact for $\delta' < \delta$ because the condition $\delta' < \delta$ ensures that these traveling bumps converge to 0 in $W^{k-1,p}_\delta$.

3.3 Properties of H^s based weighted Sobolev spaces

In [Tr76a][Tr76b] Triebel introduced a family of weighted spaces that generalize the spaces $W^{k,p}_\delta$ we have already seen. We recall the Sobolev spaces for $s \in \mathbb{R}$ and $p \in (1,\infty)$

$$H^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^* : ||u||_{H^{s,p}_\delta(\mathbb{R}^n)} = ||\mathcal{F}^{-1}\langle\xi\rangle^s \mathcal{F}u||_{L^p} < \infty \right\},$$

where $\mathcal{F}$ is the Fourier transform. For k a non-negative integer, $H^{k,p} = W^{k,p}$. We also have the more familiar Hilbert spaces $H^s = H^{s,2}$. When s is not an integer, these spaces do not have nice localization properties enjoyed by the spaces $W^{k,p}$, and this makes defining the weighted versions of these spaces somewhat more technical. The key is a variation of the rescaled norm (3.1).

Let ϕ_0 be a cutoff function equal to 1 on a neighbourhood of B_1 and supported in B_2. We define $\phi = \phi_0 - \mathcal{S}_2\phi_0$ and for $j \geq 1$ $\phi_j = \mathcal{S}_{2^{-j}}\phi$. It is easy to see that when $j \geq 1$, ϕ_j is supported in $B_{2^{j+1}} \setminus \overline{B_{2^{j-1}}}$ and that

$$\sum_{k=0}^{N} \phi_k = \mathcal{S}_{2^{-k}}\phi_0.$$

Hence $\{\phi_j\}_{j=0}^{\infty}$ is a partition of unity subordinate to a cover of $\mathbb{R}^n$ by B_2 and a collection of rescaled annuli. This type of partition of unity is commonly used in Littlewood-Payley theory.

We now define

$$H^{s,p}_\delta(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^* : ||u||^p_{H^{s,p}_\delta} = \sum_{j=0}^{\infty} 2^{-p\delta j} ||\mathcal{S}_{2^j}(\phi_j u)||^p_{H^{s,p}} < \infty \right\}$$
$$H^s_\delta(\mathbb{R}^n) = H^{s,2}_\delta(\mathbb{R}^n).$$

We have chosen to use this norm, from the several norms that Triebel showed were equivalent on these spaces, for its usefulness in computations. Its relationship to the norm $\tilde{W}^{k,p}$ is readily apparent. We note that our convention for the growth parameter δ follows Bartnik's for the spaces $W^{k,p}_\delta$ and is different from Triebel's. Our spaces $H^{s,p}_\delta$ correspond with the spaces $h^{s,p}_{2,ps-p\delta-n}$ in [Tr76a]. For the most part, we restrict our attention to the spaces H^s_δ. We will, however, occasionally use the spaces $H^{s,p}_\delta$ when s is an integer as endpoints for interpolation.

As with the unweighted Sobolev spaces, interpolation plays a fundamental role in working with spaces with a non-integral number of derivatives. Given a pair of Banach space A_0 and A_1 and a number $\theta \in (0,1)$, complex interpolation yields a Banach space $[A_0, A_1]_\theta$ with the following properties.

1. $A_1 \cap A_0 \subset [A_0, A_1]_\theta \subset A_0 + A_1$.

2. $[A_0, A_1]_\theta = [A_1, A_0]_{1-\theta}$.

3. If L is a linear map from $A_0 + A_1$ to $B_0 + B_1$ that restricts to a continuous linear map $L_i : A_i \to B_i$ for $i = 0$ and 1, then L also restricts to a continuous linear map $L_\theta : [A_0, A_1]_\theta \to [B_0, B_1]_\theta$. Moreover, $||L_\theta|| \leq ||L_0||^{1-\theta}||L_1||^\theta$.

3. If $A_1 \subset A_0$, $B_1 \subset B_0$, and $C_1 \subset C_0$, and if L_0 is a continuous bilinear map $L_0 : A_0 \times B_0 \to C_0$ that restricts to a continuous bilinear map $L_1 : A_1 \times B_1 \to C_1$, then L_0 also restricts to a continuous bilinear map $L_\theta : [A_0, A_1]_\theta \times [B_0, B_1]_\theta \to [C_0, C_1]_\theta$. Moreover, $||L_\theta|| \leq ||L_0||^{1-\theta}||L_1||^\theta$.

For a comprehensive discussion of interpolation functors, both real and complex, the reader is referred to [Tr95]. The weighted Sobolev spaces have a natural interpolation property, which we list below along with other basic facts similar to those in Lemma 3.2.

Lemma 3.3

1. *When k is a non-negative integer, an equivalent norm for $H^{k,p}_\delta$ is the norm on $W^{k,2}_\delta$.*

2. *If $\theta \in (0, 1)$, $s = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\delta = (1-\theta)\delta_0 + \theta\delta_1$, then $H^{s,p}_\delta$ is the interpolation space $[H^{s_0,p_0}_{\delta_0}, H^{s_1,p_1}_{\delta_1}]_\theta$.*

3. *H^s_δ is the dual space of $H^{-s}_{-n-\delta}$.*

4. *If $s \geq s'$ and $\delta \leq \delta'$ then H^s_δ is continuously embedded in $H^{s'}_{\delta'}$. If $s > s'$ and $\delta < \delta'$, then this embedding is compact.*

5. *If $s < n/2$, then H^s_δ is continuously embedded in L^q_δ, for every q with $\frac{1}{2} - \frac{s}{n} \leq \frac{1}{q} \leq \frac{1}{2}$.*

6. *If $s = n/2$, then H^s_δ is continuously embedded in L^q_δ, for every $q \geq 2$.*

7. *If $s > n/2$ then $H^s_\delta(\mathbb{R}^n)$ is continuously embedded in $C^0_\delta(\mathbb{R}^n)$, where $||f||_{C^0_\delta} = \sup_{x\in\mathbb{R}^n}(1 + |x|)^{-\delta}\,|f|$.*

8. *If $u \in H^s_\delta$, then $\partial u \in H^{s-1}_{\delta-1}$.*

Proof: All these claims follow immediately from [Tr76a] [Tr76b] except for claims 7, 8 and the compact embedding $H^s_\delta \hookrightarrow H^{s'}_{\delta'}$ when $s > s'$ and $\delta < \delta'$.

To prove claim 7, we know from [Tr76a] that H^s_δ embedded in $W^{k,p}_\delta$ for k an integer with $k < s$ and $k > n/p$. The claim now follows from the corresponding fact for $W^{k,p}_\delta$ spaces. Claim 8 follows easily from the corresponding property of the $W^{k,p}_\delta$ spaces, interpolation, and duality.

We now turn to the compactness argument. Let $\{u_k\}_{k=1}^\infty$ be a sequence in H^s_δ with $||u_k||_{H^s_\delta} \leq 1$. Then the sequence $\{\phi_0 u_k\}_{k=1}^\infty$ is bounded in H^s and each element has support contained in the ball B_2. From Rellich's theorem, we infer the existence of a subsequence $\{u^0_k\}_{k=0}^\infty$ such that $\{\phi_0 u^0_k\}_{k=1}^\infty$ is Cauchy in $H^{s'}$ and such that $||\chi_0(u^0_k - u^0_l)||_{H^{s'}} \leq 1$ for all $k, l \geq 1$. Similarly , the sequence $\{\mathcal{S}_{\mathcal{S}^1_2}(\phi_1 u^0_k)\}_{k=1}^\infty$ is bounded in H^s and has uniformly compact support. So there is a sub-subsequence $\{u^1_k\}_{k=1}^\infty$ such that $\{\mathcal{S}_2(\phi_1 u^1_k)\}_{k=1}^\infty$ is Cauchy in $H^{s'}$ and

$$\sum_{j=0}^{1} 2^{-2\delta' j} ||\mathcal{S}_{2^j}(\phi_j(u^1_k - u^1_l))||^2_{H^{s'}} < \frac{1}{2}$$

for $k, l \geq 1$. Continuing iteratively, we obtain sub-subsequences $\{u^m_k\}_{k=1}^\infty$ such that for $k, l \geq 1$,

$$\sum_{j=0}^{m} 2^{-2\delta' j} ||\mathcal{S}_{2^j}(\phi_j(u^m_k - u^m_l)||^2_{H^{s'}} < \frac{1}{2^m}$$

Let $v_k = u^k_1$. Then if $k, l \geq N$, since $\delta' > \delta$,

$$\begin{aligned}
||v_k - v_l||^2_{H^{s'}_{\delta'}} &= \sum_{j=0}^{\infty} 2^{-2\delta' j} ||\mathcal{S}_{2^j}(\phi_j(v_k - v_l))||^2_{H^{s'}} \\
&= \sum_{j=0}^{N} 2^{-2\delta' j} ||\mathcal{S}_{2^j}(\phi_j(v_k - v_l))||^2_{H^{s'}} + \sum_{j=N+1}^{\infty} 2^{-2\delta' j} ||\mathcal{S}_{2^j}(\phi_j(v_k - v_l))||^2_{H^{s'}} \\
&\leq 2^{-N} + 2^{-2(\delta'-\delta)(N+1)} \sum_{j=N+1}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}(\phi_j(v_k - v_l))||^2_{H^s} \\
&\leq 2^{-N} + 2^{-2(\delta'-\delta)(N+1)} ||v_k - v_l||^2_{H^s_\delta}.
\end{aligned}$$

Since the sequence $\{v_k\}_{k=1}^\infty$ is bounded in H^s_δ and since $\delta' > \delta$, we conclude the sequence is Cauchy in $H^{s'}_{\delta'}$, which proves the result. □

Since the differential operators we consider have coefficients belonging to weighted Sobolev spaces, we need to know multiplication rules for functions in these spaces. The first step is to know how to multiply functions in H^s. Although the following result is elementary, we prove it here because it is fundamental.

Lemma 3.4 *Suppose* $s_3 \leq \min(s_1, s_2)$*,* $s_1 + s_2 \geq 0$*, and* $s_3 < s_1 + s_2 - \frac{n}{2}$*. Then pointwise multiplication extends to a continuous bilinear map*

$$H^{s_1}(\mathbb{R}^n) \times H^{s_2}(\mathbb{R}^n) \to H^{s_3}(\mathbb{R}^n).$$

Proof: We first show that when $s > n/2$, then H^s is an algebra. The proof starts from the well known fact that $W^{k,p}$ is an algebra when $k > n/p$ (this is an easy consequence of Sobolev embedding). Let $k = [s]$ (i.e. k is the largest integer with $k \leq s$) and $\theta = s - k$. If $\theta = 0$, the result follows since $H^s = W^{s,2}$. Otherwise, let

$$\begin{aligned} \frac{1}{p} &= \frac{1}{2} - \frac{\theta}{n} \\ \frac{1}{q} &= \frac{1}{2} + \frac{1-\theta}{n}. \end{aligned}$$

Since $n \geq 2$ and $\theta \in (0,1)$ we have $1 < q < 2 < p < \infty$. Moreover,

$$\frac{1}{p} - \frac{k+1}{n} = \frac{1}{q} - \frac{k}{n} = \frac{1}{2} - \frac{s}{n} < 0.$$

So multiplication is a continuous bilinear map

$$W^{k+1,p} \times W^{k+1,p} \to W^{k+1,p}$$

$$W^{k,q} \times W^{k,q} \to W^{k,q}.$$

Since $k(1-\theta) + (k+1)\theta = k + \theta = s$, and since

$$\frac{\theta}{q} + \frac{1-\theta}{p} = \frac{1}{2}$$

we have $[W^{k,q}, W^{k+1,p}]_\theta = H^s$. It follows from interpolation that multiplication extends to a continuous bilinear form

$$H^s \times H^s \to H^s.$$

Since $H^s \subset C^0$ we also have a continuous bilinear map

$$H^s \times H^0 \to H^0,$$

and from interpolation

$$H^s \times H^t \to H^t$$

for all $t \in [0, s]$. We have therefore proved the result when $s_3 \geq 0$ and when either $s_1 > n/2$ or $s_2 > n/2$.

Now multiplication takes $H^0 \times H^0 \to L^1$. Since L^1 acts continuously on C^0, and since $H^s \subset C^0$ when $s > n/2$, it follows that multiplication takes

$$H^0 \times H^0 \to H^{-\frac{n}{2}-\epsilon}$$

for every $\epsilon > 0$. Interpolating between the maps

$$H^{\frac{n}{2}+\epsilon} \times H^0 \to H^0$$
$$H^0 \times H^0 \to H^{-\frac{n}{2}-\epsilon}$$

we obtain a continuous map

$$H^t \times H^0 \to H^{t-\frac{n}{2}-\epsilon}$$

for every $t \in [0, \frac{n}{2}]$ and $\epsilon > 0$. Finally, interpolating between

$$H^{\frac{n}{2}+\epsilon} \times H^t \to H^t$$
$$H^0 \times H^t \to H^{t-\frac{n}{2}-\epsilon}$$

we obtain

$$H^s \times H^t \to H^{s+t-\frac{n}{2}-\epsilon}$$

for $s, t \in [0, \frac{n}{2}]$ and $\epsilon > 0$. Letting $s_1 = s$ and $s_2 = t$, we have now proved the lemma in the case $s_1 \geq 0$ and $s_2 \geq 0$.

We now turn to the case $s_1 < 0$, which we treat by duality. If $u \in H^{s_1}_\delta$ and $v, \phi \in C^\infty_{\mathrm{c}}$, then

$$|\langle uv, \phi \rangle| = |\langle u, v\phi \rangle| \leq ||u||_{H^{s_1}} ||v\phi||_{H^{-s_1}}$$

We want to apply the previous result to estimate

$$||v\phi||_{H^{-s_1}} \lesssim ||v||_{H^{s_2}} ||\phi||_{H^{-s_3}}.$$

This requires s_2, $-s_3$, and $-s_1$ are all non-negative, $s_2 - s_3 > -s_1 - \frac{n}{2}$, and $-s_1 \leq \min(s_2, -s_3)$. It is easy to verify that all these conditions hold under the hypotheses when $s_1 < 0$, and we find

$$|\langle uv, \phi \rangle| \lesssim ||u||_{H^{s_1}} ||v||_{H^{s_2}} ||\phi||_{H^{-s_3}}.$$

Hence multiplication extends to a continuous bilinear map in this case also. The alternate case $s_2 < 0$ follows from symmetry and we are done.

$\square$

A similar multiplication rule holds for weighted spaces, taking into account the the additive behaviour of the decay parameter δ.

Lemma 3.5 *Suppose $s_3 \leq \min(s_1, s_2)$, $s_1 + s_2 \geq 0$, and $s_3 < s_1 + s_2 - \frac{n}{2}$. For any $\delta_1, \delta_2 \in \mathbb{R}$, pointwise multiplication extends to a continuous bilinear map*

$$H^{s_1}_{\delta_1}(\mathbb{R}^n) \times H^{s_2}_{\delta_2}(\mathbb{R}^n) \to H^{s_3}_{\delta_1+\delta_2}(\mathbb{R}^n).$$

Proof: Suppose $u_i \in H^{s_i}_{\delta_i}$. Taking $\phi_k = 0$ for $k < 0$ we have

$$\mathcal{S}_{2^j}(\phi_j u_1 u_2) = \mathcal{S}_{2^j}(\phi_j) \sum_{k=j-1}^{j+1} \mathcal{S}_{2^j}(\phi_k u_1) \sum_{l=j-1}^{j+1} \mathcal{S}_{2^j}(\phi_l u_2).$$

From the restrictions on s, s_1, and s_2 we know from Lemma 3.4 that multiplication is a continuous bilinear map on the corresponding unweighted Sobolev spaces. Noting that $\mathcal{S}_{2^j}\phi_j = \mathcal{S}_{2^k}\phi_k$ for $j, k \geq 1$, we find

$$\begin{aligned} ||\mathcal{S}_{2^j}(\phi_j u_1 u_2)||^2_{H^s} &\lesssim \sum_{k,l=j-1}^{j+1} ||\mathcal{S}_{2^j}(\phi_k u_1)||^2_{H^{s_1}} ||\mathcal{S}_{2^j}(\phi_l u_2)||^2_{H^{s_2}} \\ &\lesssim \sum_{k,l=j-1}^{j+1} ||\mathcal{S}_{2^{j-k}}\mathcal{S}_{2^k}(\phi_k u_1)||^2_{H^{s_1}} ||\mathcal{S}_{2^{j-l}}\mathcal{S}_{2^l}(\phi_l u_2)||^2_{H^{s_2}}. \end{aligned}$$

Now $\mathcal{S}_{2^{j-k}}$ must be one of $\mathcal{S}_{2^{-1}}$, $\mathcal{S}_{2^0}$, or $\mathcal{S}_{2^1}$, and a same result holds for $\mathcal{S}_{2^{j-l}}$. These operators are independent of j and we find

$$||\mathcal{S}_{2^j}(\phi_j u_1 u_2)||^2_{H^s} \lesssim \sum_{k,l=j-1}^{j+1} ||\mathcal{S}_{2^k}(\phi_k u_1)||^2_{H^{s_1}} ||\mathcal{S}_{2^l}(\phi_l u_2)||^2_{H^{s_2}}.$$

It follows that

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}(\phi_j u_1 u_2)||^2_{H^s} &\lesssim \sum_{j=0}^{\infty} 2^{-2\delta j} \sum_{k,l=j-1}^{j+1} ||\mathcal{S}_{2^k}(\phi_k u_1)||^2_{H^{s_1}} ||\mathcal{S}_{2^l}(\phi_l u_2)||^2_{H^{s_2}} \\ &\lesssim \sum_{k=-1}^{1} \sum_{j=0}^{\infty} \Big[2^{-2\delta_1 j} ||\mathcal{S}_{2^j}(\phi_j u_1)||^2_{H^{s_1}} \times \\ &\qquad\qquad \times\, 2^{-2\delta_2 (j+k)} ||\mathcal{S}_{2^{j+k}}(\phi_{j+k} u_2)||^2_{H^{s_2}} \Big] \\ &\lesssim \sum_{j=0}^{\infty} 2^{-2\delta_1 j} ||\mathcal{S}_{2^j}(\phi_j u_1)||^2_{H^{s_1}} \sum_{k=0}^{\infty} 2^{-2\delta_2 k} ||\mathcal{S}_{2^k}(\phi_k u_2)||^2_{H^{s_2}}. \end{aligned}$$

This proves $||u_1 u_2||_{H^s_{\delta_1+\delta_2}} \lesssim ||u_1||_{H^{s_1}_{\delta_1}} ||u_2||_{H^{s_2}_{\delta_2}}$. $\square$

We will be working with semilinear equations of the form

$$-\Delta\, u = f(x)F(u)$$

where f and u belong to weighted spaces and F is smooth. If $u \in H^s_\rho$ with $s > n/2$, then u is continuous and so is $F(u)$. We should also expect that $F(u)$ is just as regular as u, so $F(u) \in H^s_{\mathrm{loc}}$. Since $s > n/2$, multiplication by functions in H^s_ρ does not alter the regularity of functions in H^t_δ for $t \in [-s, s]$. So we would expect that under suitable restrictions, multiplication by $F(u)$ should not alter the smoothness of $f(x)$. Moreover, if $\rho < 0$, then $F(u) \to F(0)$ near infinity. So multiplication by $F(u)$ should also not alter the order of growth of $f(x)$. The following Lemma shows that these expectations are correct.

Lemma 3.6 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth. If $u \in H^s_\rho$ with $s > n/2$ and $\rho < 0$, and if $v \in H^\sigma_\delta$ with $\sigma \in [-s, s]$ and $\delta \in \mathbb{R}$, then*

$$f(u)v \in H^\sigma_\delta.$$

Moreover, the map taking (u, v) to $f(u)v$ is continuous from $H^s_\rho \times H^\sigma_\delta$ to H^σ_δ.

Proof: It is easy to verify that if $u \in H^s$ with $s > n/2$, and if η is smooth and compactly supported, then $\eta f(u) \in H^s$ and the map taking u to $\eta f(u)$ is continuous from H^s to H^s.

Now suppose u and v satisfy the hypotheses of the lemma. Then

$$\begin{aligned}
||f(u)v||^2_{H^\sigma_\delta} &= \sum_{j=0}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}(\phi_j f(u)v)||^2_{H^\sigma} \\
&= \sum_{j=0}^{\infty} 2^{-2\delta j} \left|\left| \sum_{k=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_k) f\left(\mathcal{S}_{2^j} \sum_{i=j-1}^{j+1} \phi_i u\right) \mathcal{S}_{2^j}\phi_j v \right|\right|^2_{H^\sigma} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-2\delta j} \left|\left| \sum_{k=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_k) f\left(\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i u\right) \right|\right|^2_{H^s} ||\mathcal{S}_{2^j}\phi_j v||^2_{H^\sigma}.
\end{aligned}$$

Let $\eta = \sum_{k=0}^{2} \mathcal{S}_{2^k}\phi_1$, so that for $j > 1$ we have $\sum_{k=j-1}^{j+1}(\mathcal{S}_{2^j}\phi_k) = \eta$. Since $\rho > 0$, $\mathcal{S}_{2^i}\phi_i u$ converges to 0 in H^s. It follows that $\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i u$ converges to 0 in H^s as well. Hence $\eta f\left(\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i u\right)$ converges in H^s to $\eta f(0)$, and we conclude that there exists a bound M such that

$$\left|\left| \sum_{k=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_k) f\left(\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i u\right) \right|\right|_{H^s} \leq M \tag{3.5}$$

for all $j > 1$. The cases $j = 0$ and $j = 1$ can be treated similarly. We conclude, taking M sufficiently large, that (3.5) holds for all $j \geq 0$. Hence

$$||f(u)v||^2_{H^\sigma_\delta} \lesssim M^2 ||v||^2_{H^\sigma_\delta}. \tag{3.6}$$

This proves that $f(u)v \in H^\sigma_\delta$.

To establish the continuity of the map $(u, v) \mapsto f(u)v$ acting on $H^s_\rho \times H^\sigma_\delta$, we consider any sequence $\{u_k, v_k\}_{k=1}^\infty$ converging to (u, v). Then

$$f(u)v - f(u_k)v_k = f(u)(v - v_k) - (f(u) - f(u_k))v_k.$$

From (3.6) we see that $f(u)(v - v_k) \to 0$ in H^σ_δ. We wish to establish $(f(u) - f(u_k))v_k \to 0$ as well.

Computing as before we find

$$\begin{aligned} ||(f(u) - f(u_k))v_k||^2_{H^\sigma_\delta} &\lesssim \sum_{j=0}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}\phi_j v_k||^2_{H^\sigma} \left|\left| \sum_{l=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_l)\left[f\left(R_j u\right) - f\left(R_j u_k\right)\right] \right|\right|^2_{H^s} \\ &\lesssim ||v_k||_{H^\sigma_\delta} \sup_{j\geq 0} \left|\left| \sum_{l=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_l)\left[f\left(R_j u\right) - f\left(R_j u_k\right)\right] \right|\right|^2_{H^s}, \end{aligned}$$

where $R_j w = \sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i w$. Since $\{v_k\}_{k=1}^\infty$ is bounded in H^σ_δ, it is enough to show

$$\sup_{j\geq 0} \left|\left| \sum_{l=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_l)\left[f\left(R_j u\right) - f\left(R_j u_k\right)\right] \right|\right|^2_{H^s} \to 0 \tag{3.7}$$

as $k \to \infty$.

Let $\eta = \sum_{k=0}^{2} \mathcal{S}_{2^k}\phi_1$, so that for $j > 1$ we have $\sum_{l=j-1}^{j+1}(\mathcal{S}_{2^j}\phi_l) = \eta$. The map taking u to $\eta f(u)$ is continuous from H^s to H^s. So for any fixed $\epsilon > 0$ there exists an $\alpha > 0$ such that if $||u||_{H^s} < \alpha$, then $||\eta f(u) - \eta f(0)||_{H^s} < \epsilon$. Now

$$||\eta\left[f\left(R_j u\right) - f\left(R_j u_k\right)\right]||_{H^s} \leq ||\eta f\left(R_j u\right) - \eta f(0)||_{H^s} + ||\eta f\left(R_j u_k\right) - \eta f(0)||_{H^s}. \tag{3.8}$$

To handle the first term on the right-hand side of (3.8) we note that since $\rho > 0$, $\mathcal{S}_{2^i}\phi_i u \to 0$ in H^s. Hence there is an $N > 1$ such that if $j \geq N$, then

$$||R_j u||_{H^s} = ||\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i u||_{H^s} < \frac{\alpha}{2}.$$

Consequently, for $j \geq N$,

$$||\eta f(R_j u) - \eta f(0)||_{H^s} < \epsilon. \tag{3.9}$$

We now turn to the second term of the right-hand side of (3.8). Since $\rho > 0$, we have $||\mathcal{S}_{2^i}\phi_i(u_k - u)||_{H^s} \leq ||u_k - u||_{H^s_\rho}$ for every i. Taking k sufficiently large we can therefore ensure that

$$||R_j(u_k - u)||_{H^s} = ||\sum_{i=j-1}^{j+1} \mathcal{S}_{2^{j-i}}\mathcal{S}_{2^i}\phi_i(u_k - u)||_{H^s} < \frac{\alpha}{2}$$

for every $j > 0$. We obtain for $j \geq N$ and k sufficiently large

$$\begin{aligned} ||R_j u_k||_{H^s} &\leq ||R_j u||_{H^s} + ||R_j(u_k - u)||_{H^s} \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \end{aligned}$$

Hence for $j \geq N$ and k sufficiently large

$$||\eta f(R_j u_k) - \eta f(0)||_{H^s} < \epsilon. \tag{3.10}$$

Combining (3.9) and (3.10) we conclude

$$\sup_{j \geq N} \left|\left| \sum_{l=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_l)\left[f(R_j u) - f(R_j u_k)\right] \right|\right|_{H^s} \leq 2\epsilon$$

for k sufficiently large. On the other hand,

$$\sup_{j=0}^{N-1} \left|\left| \sum_{l=j-1}^{j+1} (\mathcal{S}_{2^j}\phi_l)\left[f(R_j u) - f(R_j u_k)\right] \right|\right|_{H^s}$$

can be made as small as we please by taking k sufficiently large, since each of the finitely many terms in the supremum tends to 0 as k goes to infinity. We have hence established (3.7) and therefore also the desired continuity. $\square$

Since $1 \in H^s_\epsilon$ for every $\epsilon > 0$ we have from Lemma 3.6 that if $u \in H^s_\rho$ with $s > n/2$ and $\rho < 0$, then $f(u) = f(u) \cdot 1 \in H^s_\epsilon$ for every $\epsilon > 0$. The following corollary shows that if $f(0) = 0$ we can say more.

Corollary 3.7 *Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth and $f(0) = 0$. If $u \in H^s_\rho$ with $s > n/2$ and $\rho < 0$ then $f(u) \in H^s_\rho$ and the map taking u to $f(u)$ is continuous from H^s_ρ to H^s_ρ.*

Proof: Since $f(0) = 0$ we have from Taylor's theorem that $f(x) = F(x)x$ where F is smooth. Hence $f(u) = F(u)u \in H^s_\rho$ by Lemma 3.6, and the continuity of the map on H^s_ρ follows similarly. $\square$

Remark 1 *In practice we will use an obvious improvement to Lemma 3.6 and Corollary 3.7. It is easy to see that if f is only smooth on an open interval I, and if $[\inf u, \sup u] \subset I$, then $f(u)v \in H^\sigma_\delta$ and the map $(u,v) \mapsto f(u)v$ is continuous on $U \times H^\sigma_\delta$ for some neighbourhood U of u. An analogous statement for Corollary 3.7 also holds.*

3.4 Asymptotically Euclidean Manifolds

An asymptotically Euclidean manifold is a non-compact Riemannian manifold that can be decomposed into a compact core and a finite number of ends $\{N_i\}_{i=1}^m$. Each end N_i is diffeomorphic to the region exterior to the closed unit ball in $\mathbb{R}^n$, and the metric on N_i is asymptotic to the Euclidean metric at far distances. This loose description is made precise using weighted function spaces.

Definition 3.8 *Let M^n be a smooth, connected, n-dimensional manifold, possibly with boundary, and let g be a metric on M for which (M,g) is complete. Let E_r be the exterior region $\{x \in \mathbb{R}^n : |x| > r\}$. For $s > n/2$ and $\rho < 0$, we say (M,g) is asymptotically Euclidean (AE) of class H^s_ρ if*

1. *The metric $g \in H^s_{\mathrm{loc}}(M)$.*
2. *There exists a finite collection of charts $\{(U_i, \Phi_i\}_{i=1}^m$ on M such that $\Phi_i(U_i) = E_1$ and such that $M - \cup_i U_i$ is compact.*
3. *For each i, and for any smooth function η supported in E_1 and equal to 1 on E_2,*
$$\eta\left[(\Phi_i^{-1})^* g - \overline{g}\right] \in H^s_\rho(\mathbb{R}^n),$$
where $\overline{g}$ is the Euclidean metric.

The charts Φ_i are called end charts and the corresponding coordinates are end coordinates. Suppose (M^n, g) is asymptotically Euclidean, and let $\{(U_i, \Phi_i)\}_{i=1}^m$ be its collection of end charts. We extend this set to an atlas $\{(U_i, \Phi_i)\}_{i=1}^k$ such that for $i > m$ the set $\overline{U}_i$ is compact. Let $\{\eta_i\}_{i=1}^k$ be a partition of unity subordinate to the cover $\{U_i\}_{i=1}^k$, and for $i > m$ let V_i be $\mathbb{R}^n$ or $\mathbb{R}^n_+$ depending on whether (U_i, Φ_i) is an interior or boundary chart. The weighted Sobolev space $H^s_\delta(M)$ is the subset of $H^s_{\mathrm{loc}}(M)$ such that the norm

$$||u||_{H^s_\delta(M)} = \sum_{i=1}^m ||(\Phi_i^-1)^*(\eta_i u)||_{H^s_\delta(\mathbb{R}^n)} + \sum_{i=m+1}^k ||(\Phi_i^{-1})^* \eta_i u||_{H^s(V_i)}$$

is finite. We have a similar definition for sections of vector bundles and also for weighted $W^{k,p}$ spaces and Hölder spaces. We define define $\mathring{H}^s_\delta(M)$ to be the closure of $C^\infty_{\mathrm{c}}(\operatorname{int} M)$ in $H^s_\delta(M)$.

We can now define an AE data set (M, g, K) for the initial value problem. This is an AE manifold (M^3, g) of class H^s_ρ with $s > 3/2$ and $\rho < 0$ and a symmetric $(0, 2)$-tensor $K \in H^{s-1}_{\rho-1}$. The choice of function space for K comes from the fact that K should behave like the first derivative of the metric.

There are no surprises in translating most facts about $H^s_\delta(\mathbb{R}^n)$ to $H^s_\delta(M)$. The proof of the following Lemma follows from the corresponding results for weighted and unweighted Sobolev spaces on $\mathbb{R}^n$ and partition of unity arguments. We omit the proof.

Lemma 3.9 *Lemma 3.3 items 2, 4, 5, 6, and 7 concerning interpolation and embedding as well as Lemmas 3.5 and 3.6 and Corollary 3.7 concerning multiplication extend to AE manifolds with boundary by replacing $\mathbb{R}^n$ with M in their statements.*

We do need to be careful about generalizing the duality

$$H^\sigma_\delta(\mathbb{R}^n) = \left(H^{-\sigma}_{-n-\delta}(\mathbb{R}^n)\right)^*$$

to AE manifolds with boundary. First, since there is a boundary, the most we can hope for is

$$H^\sigma_\delta(M) = \left(\mathring{H}^{-\sigma}_{-n-\delta}(M)\right)^*.$$

But even for compact manifolds with boundary, we require a smooth metric (or at least a volume form) to make a natural identification between $H^\sigma_\delta(M)$ and $\left(\mathring{H}^{-\sigma}_{-n-\delta}(M)\right)^*$. Using the charts and partition of unity used to define the norm $H^s_\delta(M)$, we can put a smooth AE background metric $\hat{g}$ on M. We then have a bilinear form defined for $u, v \in C^\infty_{\mathrm{c}}(\operatorname{int} M)$ by

$$\langle u, v\rangle_{(M,\hat{g})} = \int_M uv\, dV_{\hat{g}}. \tag{3.11}$$

Proceeding as for compact manifolds with boundary, the bilinear form (3.11) extends to a continuous bilinear map

$$\langle \cdot, \cdot\rangle_{(M,\hat{g})} : H^\sigma_\delta \times \mathring{H}^{-\sigma}_{-n-\delta}(M) \to \mathbb{R}$$

and induces an isomorphism

$$H^\sigma_\delta(M) = \left(\mathring{H}^{-\sigma}_{-n-\delta}(M)\right)^*.$$

In applications, however, we would rather use the rough metric g instead of the smooth metric $\hat{g}$ to induce this isomorphism. This can be done on a restricted range of Sobolev spaces. Since $g - \hat{g} \in H^s_\rho$ in end coordinates, we have $dV_g = f\ dV_{\hat{g}}$ where $f > 0$ and $(f-1) \in H^s_\rho(M)$. Multiplication by f is continuous on on H^σ_δ for every σ with $|\sigma| \leq s$, so we can define for $u \in H^\sigma_\delta$ and $v \in \mathring{H}^{-\sigma}_{-n-\delta}$

$$\langle u, v \rangle_{(M,g)} = \langle fu, v \rangle_{(M,\hat{g})} .$$

For smooth functions u and v we have

$$\langle u, v \rangle_{(M,g)} = \int_M uv\ dV_g$$

and hence this definition is independent of $\hat{g}$. We have therefore established the following.

Lemma 3.10 *Suppose (M^n, g) is AE of class H^s_δ with $s > n/2$ and $\delta < 0$. If $|\sigma| \leq s$ then $H^\sigma_\delta(M)$ is naturally isomorphic to $\left(\mathring{H}^{-\sigma}_{-n-\delta}(M)\right)^*$ through the pairing $\langle \cdot, \cdot \rangle_{(M,g)}$.*

Chapter 4

LINEAR THEORY

The Lichnerowicz equation with its boundary condition is

$$\begin{aligned} -8\Delta\,\phi + R\phi - |\sigma|^2\,\phi^{-7} &= 0 \\ 4\partial_\nu\phi + H\phi - \phi^{-3}\sigma(\nu,\nu) &= 0 \quad \text{on } \partial M. \end{aligned} \tag{4.1}$$

We will solve (4.1) by means of an iterative method using the associated linear boundary value problem

$$\begin{aligned} -\Delta\,\phi + V\phi &= F \\ \partial_\nu\phi + \mu\phi &= f \quad \text{on } \partial M. \end{aligned} \tag{4.2}$$

It turns out we will also require a boundary value problem for the vector Laplacian to construct suitable transverse-traceless tensors. The Neumann boundary operator $B_{\mathbb{L}}$ for the vector Laplacian takes a vector field X to $\nu \lrcorner \mathbb{L}X = \mathbb{L}X(\nu,\cdot)$. The associated boundary value problem is then

$$\begin{aligned} \Delta_{\mathbb{L}}X &= Y \\ B_{\mathbb{L}}X &= \omega \end{aligned} \tag{4.3}$$

where ω is a 1-form over the boundary. We require a priori estimates for these boundary value problems analogous to those in Theorem 3.1 for the Euclidean Laplacian.

The mapping properties of the Laplacian of an AE metric and related linear maps have been studied extensively in the past under varying hypotheses on the regularity of the coefficients. In particular, the results of [CBC81] apply to the Laplacian of an AE metric of class H^k_ρ where k is an integer and $k > n/2 + 1$, and the results of [Ba86] apply to a metric of class $W^{2,p}_\rho$ where $p > n/2$. Boundary value problems such as (4.2) and (4.3) were treated in [LM85] for C^∞ metrics. We obtain here a technical improvement of these works that can be applied to asymptotically Euclidean metrics of class H^s_ρ with $s \in \mathbb{R}$ and $s > n/2$. The improvement is two-fold. We consider a non-integral number of derivatives, and in dimension 3 we require fewer than 2 derivatives.

Our approach is to work in local coordinates to first obtain interior estimates, then estimates on the asymptotically Euclidean ends, and finally estimates at the boundary. For notational convenience we make the following definition.

Definition 4.1 *Let A be the linear differential operator on $\mathbb{R}^n$*

$$A = \sum_{|\alpha|\leq m} a^\alpha \partial_\alpha,$$

where a^α is a $\mathbb{R}^{k\times k}$ valued function. We say that

$$A \in \mathcal{L}^{m,s}$$

if $a^\alpha \in H^{s-m+|\alpha|}$ for all $|\alpha| \leq m$. Similarly, if $\rho < 0$ we say

$$A \in \mathcal{L}^{m,s}_\rho$$

if $a^\alpha \in H^{s-m+|\alpha|}_{\rho-m+|\alpha|}$ for all $|\alpha| < m$ and if there are constant matrices a^α_∞ such that $a^\alpha_\infty - a^\alpha \in H^s_\rho$ for all $|\alpha| = m$. We call $A_\infty = \sum_{|\alpha|=m} a^\alpha_\infty \partial_\alpha$ the principal part of A at infinity.

When g is an AE metric on $\mathbb{R}^n$ of class H^s_ρ with $s > n/2$ and $\rho < 0$, then the Laplacian and vector Laplacian both belong to $\mathcal{L}^{2,s}_\rho$. From Lemmas 3.4 and 3.5 we obtain the following simple properties.

Corollary 4.2 *Suppose*

$$\begin{aligned}\eta &< \sigma + s - m - \frac{n}{2}\\ \eta &\leq \min(\sigma, s) - m\\ m &\leq \sigma + s.\end{aligned}$$

If $A \in \mathcal{L}^{m,s}$, then A is a continuous map

$$A : H^\sigma \to H^\eta.$$

If $\delta \in \mathbb{R}$, $\rho < 0$ and if $A \in \mathcal{L}^{m,s}_\rho$, then A is a continuous map

$$A : H^\sigma_\delta \to H^\eta_{\delta-m}.$$

If moreover $A_\infty = 0$, then A is a continuous map

$$A : H^\sigma_\delta \to H^\eta_{\delta-m+\rho}.$$

When $s > n/2$, the highest order coefficients of $A \in \mathcal{L}^{m,s}$ are continuous. It then makes sense to talk about their pointwise values. We say A is elliptic if for each x, the constant coefficient operator $\sum_{|\alpha|=m} a^\alpha(x)\partial_\alpha$ is elliptic. For $A \in \mathcal{L}^{m,s}_\rho$ we require additionally that A_∞ is elliptic.

4.1 Interior Estimates

For functions $u, v \in C^k$ we have a simple estimate on how the k^{th} derivatives of v contribute to the C^k norm of uv, namely $||uv||_{C^k} \lesssim ||u||_{C^0}||v||_{C^k} + ||u||_{C^k}||v||_{C^{k-1}}$. We need an analogous fact for Sobolev spaces, which will be derived from a commutator estimate for the operator $\Lambda = (1-\Delta)^{1/2}$. We start with the following estimate for an integral kernel.

Lemma 4.3 *Suppose $u \in L^2$, $s > n/2$, and $|t| \le s$. Then the kernel*

$$K_{u,s,t}(x,y) = \langle x\rangle^t \langle x-y\rangle^{-s} u(x-y) \langle y\rangle^{-t}$$

defines a continuous linear map F from L^2 to L^2 via

$$F_{u,s,t}(v)(x) = \int K_{u,s,t}(x,y)v(y)\, dV.$$

Moreover, $||F||_{L^2} \le c(s,t)||u||_{L^2}$.

Proof: We note that for smooth u and v,

$$F_{u,s,t}(v) = \mathcal{F}\Lambda^t\left(\Lambda^{-s}\mathcal{F}^{-1}u\Lambda^{-t}\mathcal{F}^{-1}v\right),$$

where $\mathcal{F}$ is the Fourier transform. Hence

$$\begin{aligned}||F_{u,s,t}(v)||_{L^2} &\le ||\Lambda^t\left(\Lambda^{-s}\mathcal{F}^{-1}u\Lambda^{-t}\mathcal{F}^{-1}v\right)||_{L^2}\\ &= ||\Lambda^{-s}\mathcal{F}^{-1}u\Lambda^{-t}\mathcal{F}^{-1}v||_{H^t}.\end{aligned}$$

Since $\Lambda^{-s}\mathcal{F}^{-1}u \in H^s$ with $s > n/2$, and since $\Lambda^{-t}\mathcal{F}^{-1}v \in H^t$ with $|t| \le s$ we can apply the multiplication rule Lemma 3.4 to obtain

$$||F_{u,s,t}(v)||_{L^2} \lesssim ||\Lambda^{-s}\mathcal{F}^{-1}u||_{H^s}||\Lambda^{-t}\mathcal{F}^{-1}v||_{H^t} = ||u||_{L^2}||v||_{L^2}$$

where the implicit constant depends on s and t but not on u or v. We can therefore extend $\mathcal{F}_{u,s,t}$ by continuity to $u \in L^2$ and $v \in L^2$ and we find $||F_{u,s,t}|| \le c(s,t)||u||$. □

We now turn to a commutator estimate for $[\Lambda^\sigma, u]$ where $u \in H^s$ with $s > n/2$. If u were smooth, this would be a pseudo-differential operator of order $\sigma - 1$, and would hence take H^σ to H^1 instead of H^0. The limited smoothness of u prevents us from gaining a full derivative of regularity, but we can show that under restricted circumstances that the commutator takes H^σ to H^θ for some $\theta \in (0, 1]$.

Since u is a pseudo-differential operator of order 0 of the low regularity type considered in [Mar88], much of the following Lemma can be deduced from the commutator estimate [Mar88] Corollary 3.4. By proving the following estimate by hand, we obtain a mildly stronger result for the particular operator of interest, and we avoid bringing in the full machinery of pseudo-differential operators with rough coefficients.

Lemma 4.4 *Suppose $u \in H^s$ with $s > n/2$, $\sigma \in [-s, s]$, and suppose $\theta \in [0, 1]$ satisfies $\theta < s - n/2$ and $\theta \leq s - \sigma$. Then $[\Lambda^\sigma, u]$ is continuous as a map*

$$[\Lambda^\sigma, u] : H^\sigma \to H^\theta.$$

Moreover,

$$||[\Lambda^\sigma, u]||_{H^\sigma} \lesssim ||u||_{H^s}$$

for all $u \in H^s$.

Proof: By applying Fourier transforms, it is enough to show that for every $U \in L^2$

$$\Lambda^\theta [\Lambda^\sigma, \Lambda^{-s} \mathcal{F}^{-1} U] \Lambda^{-\sigma} \mathcal{F}^{-1}$$

is continuous from L^2 to L^2 and $||F||_{L^2} \lesssim ||U||_{L^2}$. Upon taking Fourier transforms, we are thus lead to consider the integral kernel

$$K_U = \langle \xi \rangle^\theta \langle \xi - \zeta \rangle^{-s} U(\xi - \zeta) \left(\langle \zeta \rangle^{-\sigma} \langle \xi \rangle^\sigma - 1 \right).$$

Writing K_U in the form

$$\langle \xi - \zeta \rangle^{-s} U(\xi - \zeta) \langle \xi \rangle^\theta \langle \zeta \rangle^{-\sigma} (\langle \xi \rangle^\sigma - \langle \zeta \rangle^\sigma)$$

we wish to estimate $\langle \xi \rangle^{\sigma} - \langle \zeta \rangle^{-\sigma}$. Now

$$\begin{aligned} |\langle \xi \rangle^{\sigma} - \langle \zeta \rangle^{\sigma}| &= \left| \int_0^1 \frac{d}{dt} \langle t\xi + (1-t)\zeta \rangle^{\sigma} \ dt \right| \\ &\lesssim \left| \int_0^1 \frac{d}{dt} \langle t\xi + (1-t)\zeta \rangle^{\sigma-1} \ dt \right| \langle \xi - \zeta \rangle . \end{aligned}$$

From the monotonicity of the function $x \mapsto (1+x^2)^{\sigma}$ for $x \geq 0$ we conclude

$$|\langle \xi \rangle^{\sigma} - \langle \zeta \rangle^{\sigma}| \lesssim \left(\langle \xi \rangle^{\sigma-1} + \langle \zeta \rangle^{\sigma-1} \right) \langle \xi - \zeta \rangle .$$

Hence, for any $\theta \in [0,1]$,

$$|\langle \xi \rangle^{\sigma} - \langle \zeta \rangle^{\sigma}| \lesssim (\langle \xi \rangle^{\sigma} + \langle \zeta \rangle^{\sigma})^{1-\theta} \left(\langle \xi \rangle^{\sigma-1} + \langle \zeta \rangle^{\sigma-1} \right)^{\theta} \langle \xi - \zeta \rangle^{\theta} .$$

Now for $\theta \in [0,1]$ and $a, b \geq 0$ we have $(a+b)^{\theta} \leq a^{\theta} + b^{\theta}$. We we conclude, after an easy computation,

$$\begin{aligned} |K_U(\xi,\zeta)| \lesssim \langle \xi - \zeta \rangle^{-s+\theta} |U(\xi - \zeta)| \Big(&\langle \xi \rangle^{\sigma} \langle \zeta \rangle^{-\sigma} + \langle \xi \rangle^{(1-\theta)\sigma+\theta} \langle \zeta \rangle^{-(1-\theta)\sigma-\theta} + \\ &\langle \xi \rangle^{\theta\sigma} \langle \zeta \rangle^{-\theta\sigma} + \langle \xi \rangle^{\theta} \langle \zeta \rangle^{-\theta} \Big) . \end{aligned}$$

We want to show that the right-hand side is a sum of kernels of the form considered in Lemma 4.3. This is true so long as $s - \theta > n/2$ and additionally

$$\begin{aligned} s + \theta \ \leq \ & \quad \sigma & \leq \ s - \theta \\ -s + \theta \ \leq \ & \quad \theta & \leq \ s - \theta \\ -s + \theta \ \leq \ & (1-\theta)\sigma + \theta & \leq \ s - \theta. \end{aligned}$$

If we assume $\sigma \geq 0$, then from the relationships $s - \theta > n/2 \geq 1$, $\theta + \sigma \leq s$ and $\theta \leq 1$, all these inequalities are obvious except possibly

$$(1-\theta)\sigma + \theta \leq s - \theta. \tag{4.4}$$

If $0 \leq \sigma \leq 1$, then $(1-\theta)\sigma + \theta \in [0,1]$ and hence (4.4) holds. Otherwise, if $s \geq 1$ we have $-\theta\sigma \leq -\theta$ and therefore

$$(\sigma + \theta) - \sigma\theta \leq (\sigma + \theta) - \theta \leq s - \theta$$

as required. To handle the case $\sigma < 0$ we write K_U as

$$K_U = \langle \xi - \zeta \rangle^{-s} U(\xi - \zeta) \langle \xi \rangle^{\sigma+\theta} \left(\langle \zeta \rangle^{-\sigma} - \langle \xi \rangle^{-\sigma} \right).$$

A similar computation as before results in the estimate

$$|K_U(\xi,\zeta)| \lesssim \langle \xi - \zeta \rangle^{-s+\theta} |U(\xi - \zeta)| \Big(1 + \langle \xi \rangle^{\sigma+\theta} \langle \zeta \rangle^{-\sigma-\theta} + \langle \xi \rangle^{-(1-\theta)\sigma} \langle \zeta \rangle^{(1-\theta)\sigma} + \\ \langle \xi \rangle^{\theta(\sigma+1)} \langle \zeta \rangle^{-\theta(\sigma+1)} \Big).$$

We have the resulting restrictions

$$\begin{aligned} s + \theta \leq \quad & \sigma + \theta \quad \leq s - \theta \\ -s + \theta \leq \; & (1-\theta)\sigma \leq s - \theta \\ -s + \theta \leq \quad & \theta\sigma + \theta \quad \leq s - \theta. \end{aligned}$$

Since $s - \theta \geq n/2$, $-s \leq \sigma \leq 0$ and $0 \leq \theta \leq 1$, all these inequalities are obvious except

$$-s + \theta \leq (1-\theta)\sigma.$$

If $-1 \leq \sigma \leq 0$ then

$$-s + \theta < -\frac{n}{2} \leq -1 \leq (1-\theta)\sigma.$$

Otherwise, $\sigma \leq -1$ and $\theta \leq -\sigma\theta$. Since $-s \leq \sigma$ we obtain

$$-s + \theta \leq \sigma - \theta\sigma$$

as required.

We have dominated K_U by integral kernels that determine continuous involutions of L^2 with norms that depend only on the norm of $|U|$ in L^2. Hence K_U is also a continuous involution of L^2 and $||K_U||_{L^2} \lesssim ||U||_{L^2}$. □

Lemma 4.5 *Suppose $s > n/2$, $\sigma \in (-s, s]$, and $\theta \in (0,1]$ satisfies $\theta < \min(s - n/2, s + \sigma)$. Then for all $u \in H^s$ and $v \in H^\sigma$ we have*

$$||uv||_{H^\sigma} \lesssim ||u||_{L^\infty} ||v||_{H^\sigma} + ||u||_{H^s} ||v||_{H^{\sigma-\theta}}. \tag{4.5}$$

Proof: For all $\phi \in C_c^\infty$,

$$\begin{aligned}\langle uv, \phi\rangle &= \left\langle \Lambda^\sigma v, \Lambda^{-\sigma} u\phi\right\rangle \\ &= \left\langle \Lambda^\sigma v, u\Lambda^{-\sigma}\phi\right\rangle + \left\langle v, \Lambda^\sigma[\Lambda^{-\sigma}, u]\phi\right\rangle . \\ &\lesssim ||u||_{L^\infty}||v||_{H^\sigma}||\phi||_{H^{-\sigma}} + ||v||_{H^{\sigma-\theta}}||[\Lambda^{-\sigma}, u]\phi||_{H^\theta}.\end{aligned}$$

From Lemma 4.4,

$$||[\Lambda^{-\sigma}, u]\phi||_{H^\theta} \lesssim ||u||_{H^s}||\phi||_{H^{-\sigma}}.$$

Hence

$$||uv||_{H^\sigma} \lesssim ||u||_{L^\infty}||v||_{H^\sigma} + ||u||_{H^s}||v||_{H^{\sigma-\theta}}. \tag{4.6}$$

□

We can now prove an interior regularity a-priori estimate for elliptic operators in $\mathcal{L}^{m,s}$. We proceed with the proof in two steps working with the high order and low order terms separately.

Proposition 4.6 *Suppose $s > n/2$ and suppose $A \in \mathcal{L}^{m,s}$ is elliptic and has only has terms of order m (i.e. $A = \sum_{|\alpha|=m} a^\alpha(x)\partial_\alpha$). If $\sigma \in (m-s, s]$, then for all $u \in H^\sigma$ supported in a compact set K*

$$||u||_{H^\sigma} \lesssim ||Au||_{H^{\sigma-m}} + ||u||_{H^{m-s}}, \tag{4.7}$$

where the implicit constant depends on K and A but not on u.

Proof: Fix $x_0 \in K$ and let ϵ be a small parameter to be chosen later. Let χ be a cutoff function equal to 1 on B_1 and equal to 0 outside B_2, and let $\chi_\epsilon(x) = \chi((x-x_0)/\epsilon)$. Let $A = A_m + R$ where A_m is the constant coefficient operator $\sum_{|\alpha|=m} a^\alpha(x_0)\partial_\alpha$ and $R = \sum_{|\alpha|=m} r^\alpha\partial_\alpha = \sum_{|\alpha|=m}\left(a^\alpha - a^\alpha(x_0)\right)\partial_\alpha$.

From elliptic theory for constant coefficient elliptic operators we have

$$\begin{aligned}||\chi_\epsilon u||_{H^\sigma} &\lesssim ||A_m\chi_\epsilon u||_{H^{\sigma-m}} + ||\chi_\epsilon u||_{H^{m-s}} \\ &\lesssim ||A\chi_\epsilon u||_{H^{\sigma-m}} + ||R\chi_\epsilon u||_{H^{\sigma-m}} + ||\chi_\epsilon u||_{H^{m-s}} \\ &\lesssim c(\epsilon)||Au||_{H^{\sigma-m}} + ||[A,\chi_\epsilon]u||_{H^{\sigma-m}} + ||R\chi_\epsilon u||_{H^{\sigma-m}} + c(\epsilon)||u||_{H^{m-s}}.\end{aligned}$$

To estimate the term $||R\chi_\epsilon u||_{H^{\sigma-m}}$, we have

$$\begin{aligned}||r^\alpha\partial_\alpha\chi_\epsilon u||_{H^{\sigma-m}} &= ||\chi_{2\epsilon}r^\alpha\partial_\alpha\chi_\epsilon u||_{H^{\sigma-m}} \\ &\lesssim ||\chi_{2\epsilon}r^\alpha||_{L^\infty}||\chi_\epsilon u||_{H^\sigma} + \sum_{|\alpha|=m}||\chi_{2\epsilon}r^\alpha||_{H^s}||\chi_\epsilon u||_{H^{\sigma-\theta}},\end{aligned}$$

where $\theta > 0$ is a constant given by Lemma 4.5 satisfying $\sigma - \theta > m - s$. Taking ϵ sufficiently small we we can make $||\chi_{2\epsilon} r^\alpha||_{L^\infty}$ as small as we please and we obtain for an ϵ depending only on A and x_0

$$||\chi_\epsilon u||_{H^\sigma} \lesssim c(\epsilon)||Au||_{H^{\sigma-m}} + ||[A,\chi_\epsilon]u||_{H^{\sigma-m}} + c(\epsilon, A, x_0)||u||_{H^{\sigma-\theta}}.$$

Now $[A,\chi_\epsilon] \in \mathcal{L}^{m-1,s}$ and hence from Corollary 4.2

$$||[A,\chi_\epsilon]u||_{H^{\sigma-m}} \lesssim c(\epsilon, A, x_0)||u||_{H^{\sigma-1}}.$$

Since $\theta \le 1$, we obtain

$$||u||_{H^\sigma(B_{\epsilon/2}(x))} \lesssim c(\epsilon, A, x_0)\left[||Au||_{H^{\sigma-m}} + ||u||_{H^{\sigma-\theta}}\right].$$

Covering K with finitely many such balls we find

$$||u||_{H^\sigma} \lesssim ||Au||_{H^{\sigma-m}} + ||u||_{H^{\sigma-\theta}},$$

where the implicit constant depends on K and A. Since $\sigma - \theta > m - s$, equation (4.7) then follows from interpolation. $\square$

Proposition 4.7 *Let U and V be open sets with $U \subset\subset V$, and suppose $s > n/2$ and $\sigma \in (m-s, s]$. If $A \in \mathcal{L}^{m,s}$ is elliptic, then for every $u \in H^\sigma$ we have*

$$||u||_{H^\sigma(U)} \lesssim ||Au||_{H^{\sigma-m}(V)} + ||u||_{H^{m-s}(V)}. \tag{4.8}$$

Proof: Choose an open set V_0 such that $U \subset\subset V_0 \subset\subset V$, and let χ be a cutoff function equal to 1 on U and compactly supported in V_0. Let $A = A_m + A_{\text{low}}$ where A_m is the order m operator $\sum_{|\alpha|=m} a^\alpha \partial_\alpha$. From Proposition 4.6 we have

$$\begin{aligned}||\chi u||_{H^\sigma} &\lesssim ||A_m \chi u||_{H^{\sigma-m}} + ||\chi u||_{H^{m-s}} \\ &\lesssim ||\chi A u||_{H^{\sigma-m}} + ||[A,\chi]u||_{H^{\sigma-m}} + ||A_{\text{low}}\chi u||_{H^{\sigma-m}} + ||\chi u||_{H^{m-s}}.\end{aligned} \tag{4.9}$$

Let χ' be a second cutoff function equal to 1 on $\operatorname{supp}\chi$ and also compactly supported in V_0. Arguing as in Proposition 4.6 we have

$$||[A,\chi]u||_{H^{\sigma-m}} \lesssim ||\chi' u||_{H^{\sigma-1}}. \tag{4.10}$$

Now $A_{\text{low}} \in \mathcal{L}^{m-1,s-1}$. Pick θ such that $\theta < s - \frac{n}{2}$ and $\theta \le \min(1, \sigma - (m-s))$. Then from Corollary 4.2 we have

$$||A_{\text{low}}\chi u||_{H^{\sigma-m}} \lesssim ||\chi u||_{H^{\sigma-\theta}}. \tag{4.11}$$

Combining equations (4.9)–(4.11), we obtain

$$||u||_{H^\sigma(U)} \lesssim ||Au||_{H^{\sigma-m}(V_0)} + ||u||_{H^{\sigma-\theta}(V_0)}.$$

Finally, we obtain (4.8) by a standard iteration procedure working with an increasing sequence of open sets $U \subset\subset V_0 \subset\subset \cdots \subset\subset V_M \subset\subset V$ for some M sufficiently large depending only on $s - \frac{n}{2}$ and $\sigma - (m-s)$. □

4.2 Estimates at Infinity

The key to proving elliptic estimates on weighted spaces is the following generalization of Lemma 5.1 of [CBC81].

Lemma 4.8 *Let A be a homogeneous constant coefficient linear elliptic operator of order $m < n$ on $\mathbb{R}^n$. For $s \in \mathbb{R}$ and $\delta \in (m-n, 0)$ we have $A : H^s_\delta \to H^{s-m}_{\delta-m}$ is an isomorphism.*

Proof: We consider three ranges of s: $[m,\infty)$, $[-\infty, 0]$ and $[0, m]$.

Let $A_{s,\delta}$ denote A acting on H^s_δ. From [CBC81] we know that if k is an integer and $k \ge m$, then $A_{k,\delta}$ has an inverse $A^{-1}_{k,\delta}$. For $s \in [k, k+1]$ we find from interpolation that $A^{-1}_{k,\delta}$ restricts to a map $B_{s,\delta} : H^{s-m}_{\delta-m} \to H^s_\delta$, and it easily follows that $B_{s,\delta} = A^{-1}_{s,\delta}$. This establishes the result for $s \in [m, \infty)$.

To obtain the result for $s \in (-\infty, 0]$ we recall that $H^s_\delta = \left(H^{-s}_{-n-\delta}\right)^*$. Let A^* be the adjoint of A. From the above we know that if $s \le 0$ and if $\delta \in (m-n, 0)$, then $A^*_{-s+m,-\delta-n+m}$ is an isomorphism. For $u \in H^{s-m}_{\delta-m}$ let $B_{s,\delta}u$ be the distribution defined by

$$\langle B_{s,\delta}u, \phi\rangle = \left\langle u, (A^*_{-s+m,-\delta+m-n})^{-1}\phi\right\rangle$$

for all $\phi \in C^\infty_0$. Now

$$|\langle B_{s,\delta}u, \phi\rangle| \le ||u||_{H^{s-m}_{\delta-m}} ||(A^*_{-s+m,-\delta-n+m})^{-1}||_{H^{-s}_{-\delta-n}} ||\phi||_{H^{-s}_{-\delta-n}}.$$

This proves $B_{s,\delta}u \in H^s_\delta$ and we obtain a continuous map from $H^{s-m}_{\delta-m} \to H^s_\delta$. It easily follows from the definition of $B_{s,\delta}$ that $B_{s,\delta} = A^{-1}_{s,\delta}$. So the result holds for $s \in (-\infty, 0]$.

Finally, the result for $s \in [0, m]$ is obtained by interpolation. □

Combining Proposition 4.7 and Lemma 4.8 we have the following mapping property of elliptic operators in $\mathcal{L}^{m,s}_\rho$ on weighted spaces on $\mathbb{R}^n$. The approach of the proof is standard [Ca79b][CBC81][Ba86] with some small changes needed to accommodate the weighted H^s spaces.

Proposition 4.9 *Suppose $A \in \mathcal{L}^{m,s}_\rho$ where $s > n/2$, $\sigma \in (m-s, s]$, and $\rho < 0$. Then if $m - n < \delta < 0$, and $\delta' \in \mathbb{R}$ we have*

$$||u||_{H^\sigma_\delta} \lesssim ||Au||_{H^{\sigma-m}_{\delta-m}} + ||u||_{H^{s-m}_{\delta'}} \tag{4.12}$$

for every $u \in H^\sigma$. In particular, A is semi-Fredholm as a map from H^σ_δ to $H^{\sigma-m}_{\delta-m}$.

Proof: Let $A = A_\infty + R$ where A_∞ is the principal part of A at infinity. Let χ be a cutoff function such that $1-\chi$ has support contained in B_2 and is equal to 1 on B_1. Let r be a fixed dyadic integer to be selected later, let $\chi_r(x) = \chi(x/r)$, and let $u_r = \chi_r u$. From Lemma 4.8 we have

$$||u_r||_{H^\sigma_\delta} \lesssim ||A_\infty u_r||_{H^{\sigma-m}_{\delta-m}}.$$

Hence

$$||u_r||_{H^\sigma_\delta} \lesssim ||Au_r||_{H^{\sigma-m}_{\delta-m}} + ||Ru_r||_{H^{\sigma-m}_{\delta-m}}$$

where the implicit constant does not depend on r. Now $R \in \mathcal{L}^{m,s}_\rho$ has vanishing principal part at infinity. Hence, from Corollary 4.2 we obtain

$$||Ru_r||_{H^{\sigma-m}_{\delta-m}} \lesssim ||R||_{H^\sigma_{\delta-\rho}} ||\chi_{r/2}||_{H^s_{-\rho}} ||u_r||_{H^\sigma_\delta}$$

From Lemma 4.10 proved below we have

$$\lim_{j\to\infty} ||\chi_{2^j}||_{H^\sigma_{-\rho}} = 0.$$

Fixing r large enough we obtain

$$\begin{aligned} ||u_r||_{H^\sigma_\delta} &\lesssim ||Au_r||_{H^{\sigma-m}_{\delta-m}} \\ &\lesssim ||\chi_r Au||_{H^{\sigma-m}_{\delta-m}} + ||[A,\chi_r]u||_{H^{\sigma-m}_{\delta-m}} \\ &\lesssim ||Au||_{H^{\sigma-m}_{\delta-m}} + ||u||_{H^\sigma(B_{2r})}. \end{aligned} \tag{4.13}$$

Let $u_0 = (1-\chi_r)u$. Then $||u_0||_{H^\sigma(B_{2r})} \lesssim ||u||_{H^\sigma(B_{2r})}$ and hence

$$\begin{aligned} ||u||_{H^\sigma_\delta} &\lesssim ||u_r||_{H^\sigma_\delta} + ||u_0||_{H^\sigma_{B_{2r}}} \\ &\lesssim ||Au||_{H^{\sigma-m}_{\delta-m}} + ||u||_{H^\sigma(B_{2r})} \end{aligned}$$

From Proposition 4.7 we then obtain

$$||u||_{H^\sigma_\delta} \lesssim ||Au||_{H^{\sigma-m}_{\delta-m}} + ||u||_{H^{s-m}(B_{3r})}.$$

Equation (4.12) now follows since for each $\delta' \in \mathbb{R}$,

$$||u||_{H^{s-m}(B_{3r})} \lesssim ||u||_{H^{s-m}_{\delta'}}.$$

That A is semi-Fredholm is an immediate consequence of (4.12) choosing any $\delta' > \delta$. □

The following scaling lemma now completes the proof of Proposition 4.9.

Lemma 4.10 *Suppose $f \in H^s_\delta$ with $s \in \mathbb{R}$ and $\delta > 0$, and suppose f vanishes in a neighbourhood of the origin. Then*

$$\lim_{i\to\infty} ||\mathcal{S}_{2^{-i}} f||_{H^s_\delta} = 0.$$

Proof: Without loss of generality we can assume that f vanishes on B_2. Then $\mathcal{S}_{2^{j-i}} f = 0$ on B_2 whenever $j \le i$. So

$$\begin{aligned} ||\mathcal{S}_{2^{-i}} f||^2_{H^s_\delta} &= \sum_{j=i+1}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}(\phi_j \mathcal{S}_{2^{-i}} f)||^2_{H^s} \\ &= 2^{-2\delta i} \sum_{j=1}^{\infty} 2^{-2\delta j} ||\mathcal{S}_{2^j}(\phi_j f)||^2_{H^s} \\ &\le 2^{-2\delta i} ||f||^2_{H^s_\delta}. \end{aligned}$$

Since $\delta > 0$, the result is proved. □

4.3 Estimates on Manifolds with Boundary

For simplicity we now restrict our attention to the Neumann problems for the Laplacian and vector Laplacian. If $\Omega \subset \mathbb{R}^n$ is a bounded open subset with smooth boundary, we have the following classical estimate for a smooth metric on Ω and $s > 3/2$.

$$||u||_{H^s(\Omega)} \lesssim ||\Delta\, u||_{H^{s-2}(\Omega)} + ||\partial_\nu u||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||u||_{H^0(\Omega)} \tag{4.14}$$

This estimate is familiar when $s \geq 2$ from [Hö85], and can be proved using the theory of weak solutions (e.g. [Mc00]) and the interpolation method of [LE72]. Similarly, we have for $s > 3/2$

$$||X||_{H^s(\Omega)} \lesssim ||\Delta_{\mathbb{L}} X||_{H^{s-2}(\Omega)} + ||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||X||_{H^0(\Omega)}. \tag{4.15}$$

Extending the coefficient freezing arguments of Section 4.1 we now show we have similar estimates for rough metrics. The first step is the generalization of Lemma 4.5 to bounded domains.

Lemma 4.11 *Let Ω be a bounded domain with smooth boundary. Suppose $u \in H^s(\Omega)$ and $v \in H^\sigma(\Omega)$ with $s > n/2$, $\sigma \in (-s, s]$, and $\theta \in (0,1]$ satisfies $\theta < \min(s - n/2, s + \sigma)$. Then*

$$||uv||_{H^\sigma(\Omega)} \lesssim ||u||_{L^\infty(\Omega)} ||v||_{H^\sigma(\Omega)} + ||u||_{H^s} ||v||_{H^{\sigma-\theta}(\Omega)}. \tag{4.16}$$

Proof: Let N be an integer with $N > s$. From [Tr95] we know there exists an extension operator E taking $H^\sigma(\Omega)$ to $H^\sigma(\mathbb{R}^n)$ for all σ with $|\sigma| \leq N$ and such that

$$||E_N u||_{L^\infty(\mathbb{R}^n)} \lesssim ||E_N u||_{L^\infty(\Omega)} \tag{4.17}$$

for all $u \in L^\infty(\Omega) \cap H^\sigma(\mathbb{R}^n)$. Now restriction is a continuous map from $H^\sigma(\mathbb{R}^n)$ to $H^\sigma(\Omega)$, so

$$||uv||_{H^\sigma(\Omega)} \lesssim ||E(u)E(v)||_{H^\sigma(\mathbb{R}^n)}. \tag{4.18}$$

From Lemma 4.5 applied to the right hand side of (4.18) together with the continuity of the extension map and (4.17) we find

$$\begin{aligned} ||uv||_{H^\sigma(\Omega)} &\lesssim ||E(u)||_{L^\infty(\mathbb{R}^n)} ||E(v)||_{H^\sigma(\mathbb{R}^n)} + ||E(u)||_{H^s(\mathbb{R}^n)} ||E(v)||_{H^{\sigma-\tau}(\mathbb{R}^n)} \\ &\lesssim ||u||_{L^\infty(\Omega)} ||v||_{H^\sigma(\Omega)} + ||u||_{H^s(\Omega)} ||E(v)||_{H^{\sigma-\tau}(\Omega)} \end{aligned}$$

as required. □

Proposition 4.12 *Suppose Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary, $s > n/2$, and $g \in H^s(\Omega)$ is a metric on Ω. Then*

$$||X||_{H^s(\Omega)} \lesssim ||\Delta_{\mathbb{L}} X||_{H^{s-2}(\Omega)} + ||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||X||_{H^0(\Omega)}, \tag{4.19}$$

for all $X \in H^s(\Omega)$, and

$$||u||_{H^s(\Omega)} \lesssim ||\Delta\, u||_{H^{s-2}(\Omega)} + ||\partial_\nu u||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||u||_{H^0(\Omega)}, \tag{4.20}$$

for all $u \in H^s(\Omega)$.

Proof: We treat the case of the vector Laplacian; the proof for the scalar Laplacian proceeds identically. Fix $x_0 \in \partial\Omega$. Let χ be a cutoff function supported in B_1 and equal to 1 on $B_{\frac{1}{2}}$. Let $\chi_\epsilon(x) = \chi((x-x_0)/\epsilon)$, and let $\Delta^0_{\mathbb{L}}$ and $B^0_{\mathbb{L}}$ be the constant coefficient differential operators corresponding to the metric $g(x_0)$. Then from (4.15)

$$||\chi_\epsilon X||_{H^s(\Omega)} \lesssim ||\Delta^0_{\mathbb{L}}\chi_\epsilon X||_{H^{s-2}(\Omega)} + ||B^0_{\mathbb{L}}\chi_\epsilon X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||\chi_\epsilon X||_{H^0(\Omega)}.$$

To treat the boundary term we write

$$B^0_{\mathbb{L}}\chi_\epsilon X = \chi_\epsilon B_{\mathbb{L}} X + [B_{\mathbb{L}}, \chi_\epsilon]X + (B^0_{\mathbb{L}} - B_{\mathbb{L}})\chi_\epsilon X. \tag{4.21}$$

Now ν can be extended off the boundary as an $H^s(\Omega)$ vector field $\hat{\nu}$, and $B_{\mathbb{L}}X$ is the restriction of $\mathbb{L}(X)(\hat{\nu}, \cdot)$ to the boundary. Let $E_{\mathbb{L}}X = \mathbb{L}(X)(\hat{\nu}, \cdot)$, and let $E^0_{\mathbb{L}}$ be the constant coefficient operator corresponding to g_0. From the trace laws

$$||[B_{\mathbb{L}}, \chi_\epsilon]X||_{H^{s-\frac{3}{2}}(\partial\Omega)} \lesssim ||[E_{\mathbb{L}}, \chi_\epsilon]X||_{H^{s-1}(\Omega)}$$
$$||(B^0_{\mathbb{L}} - B_{\mathbb{L}})\chi_\epsilon X||_{H^{s-\frac{3}{2}}(\partial\Omega)} \lesssim ||(E^0_{\mathbb{L}} - E_{\mathbb{L}})\chi_\epsilon X||_{H^{s-1}(\Omega)}.$$

Now $[E_{\mathbb{L}}, \chi_\epsilon]$ is just an $H^s(\Omega)$ function and hence

$$||[E_{\mathbb{L}}, \chi_\epsilon]X||_{H^{s-1}(\Omega)} \lesssim ||[E_{\mathbb{L}}, \chi_\epsilon]||_{H^s(\Omega)}||X||_{H^{s-1}(\Omega)}. \tag{4.22}$$

On the other hand, $(E^0_{\mathbb{L}} - E_{\mathbb{L}})$ can be written as $\sum_{|\alpha|\le 1} a^\alpha \partial_\alpha$ where $a^\alpha \in H^{s-1+|\alpha|}$ and $a^\alpha(x_0) = 0$ if $|\alpha| = 1$. Fix $\theta \in (0,1]$ with $s - \frac{n}{2} < \theta$. When $|\alpha| = 1$ we obtain from Lemma 4.11

$$||a^\alpha\partial_\alpha\chi_\epsilon X||_{H^{s-1}(\Omega)} \lesssim ||\chi_{2\epsilon}a^\alpha||_{L^\infty(\Omega)}||\chi_\epsilon X||_{H^s(\Omega)} + ||\chi_{2\epsilon}a^\alpha||_{H^s(\Omega)}||\chi_\epsilon X||_{H^{s-\theta}(\Omega)}. \tag{4.23}$$

On the other hand, from the multiplication law we obtain for $|\alpha| = 0$

$$||a^\alpha\chi_\epsilon X||_{H^{s-1}(\Omega)} \lesssim ||a^\alpha||_{H^{s-1}(\Omega)}||\chi_\epsilon X||_{H^{s-\theta}(\Omega)}. \tag{4.24}$$

Combining equations (4.21), (4.22), (4.23) and (4.24) we arrive at

$$\begin{aligned}||B^0_{\mathbb{L}}\chi_\epsilon X||_{H^{s-\frac{3}{2}}(\partial\Omega)} &\lesssim c(\epsilon)||B_{\mathbb{L}}X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + \\ &\sum_{|\alpha|=1} ||\chi_{2\epsilon}a^\alpha||_{L^\infty(\Omega)}||\chi_\epsilon X||_{H^s(\Omega)} + c(\epsilon, g, x_0)||X||_{H^{s-\theta}(\Omega)}.\end{aligned} \tag{4.25}$$

On the other hand, the estimate for

$$||\Delta^0_{\mathbb{L}}\chi_\epsilon X||_{H^{s-2}(\Omega)}$$

proceeds exactly as in the case of no boundary using Lemma 4.11 in place of Lemma 4.5. We therefore obtain, taking ϵ small enough to merge the terms involving $||\chi_\epsilon X||_{H^s(\Omega)}$ into the right hand side,

$$||\chi_\epsilon X||_{H^s(\Omega)} \lesssim c(\epsilon)||\Delta_{\mathbb{L}} X||_{H^{s-2}(\Omega)} + c(\epsilon)||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + c(\epsilon, g, x_0)||X||_{H^{s-\theta}(\Omega)}.$$

Covering the boundary with finitely of the balls $B_{\frac{\epsilon}{2}}(x_0)$ and using the interior estimate Proposition 4.7 to treat the domain away from the boundary we have

$$||X||_{H^s(\Omega)} \lesssim ||\Delta_{\mathbb{L}} X||_{H^{s-2}(\Omega)} + ||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||X||_{H^{s-\theta}(\Omega)}.$$

On the other hand, since $0 < s - \theta < s$, we obtain from interpolation

$$||X||_{H^{s-\theta}(\Omega)} \le \epsilon ||X||_{H^s(\Omega)} + c(\epsilon)||X||_{H^0(\Omega)}$$

for any $\epsilon > 0$. Hence

$$||X||_{H^s(\Omega)} \lesssim ||\Delta_{\mathbb{L}} X||_{H^{s-2}(\Omega)} + ||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial\Omega)} + ||X||_{H^0(\Omega)}.$$

$\square$

From a partition of unity argument and Propositions 4.7, 4.9, and 4.12 we have now obtained the desired estimates for AE manifolds.

Proposition 4.13 *Suppose (M^n, g) is AE of class H^s_ρ and with $s > n/2$ and $\rho < 0$, and suppose $\delta \in (2-n, 0)$ and $\delta' \in \mathbb{R}$. For every vector field $X \in H^s_\delta(M)$,*

$$||X||_{H^s_\delta(M)} \lesssim ||\Delta_{\mathbb{L}} X||_{H^{s-2}_{\delta-2}(M)} + ||B_{\mathbb{L}} X||_{H^{s-\frac{3}{2}}(\partial M)} + ||X||_{H^0_{\delta'}(M)}.$$

For every function $u \in H^s_\delta(M)$,

$$||u||_{H^s_\delta(M)} \lesssim ||\Delta\, u||_{H^{s-2}_{\delta-2}(M)} + ||\partial_\nu u||_{H^{s-\frac{3}{2}}(\partial M)} + ||u||_{H^0_{\delta'}(M)}.$$

4.4 Integration by Parts

For functions u and v in $C^\infty_{\mathrm{c}}(M)$ (noting that M includes its boundary), and smooth asymptotically Euclidean metrics g, we have the formula

$$\int_M \langle \nabla u, \nabla v\rangle_g \; dV = -\int_M u\Delta\, v \; dV + \int_{\partial M} u\partial_\nu v \; dA. \tag{4.26}$$

That this formula also holds for asymptotically Euclidean metrics of class H^s_ρ with $s > n/2$ and $\rho < 0$ is clear from using an approximating sequence of smooth metrics. A little care is required, however, to reduce the regularity of u and v and to remove the hypothesis of compact support. Fixing $u \in C^\infty_{\mathrm{c}}(M)$, we first show that (4.26) holds for $v \in H^s_\delta(M)$ for any $s > 3/2$ and $\delta \in \mathbb{R}$. To see this, let $\{v_k\}_{k=1}^\infty$ be a sequence of functions in $C^\infty_{\mathrm{c}}(M)$ converging to v in H^s_δ. Then

$$\int_M \langle \nabla u, \nabla v_k\rangle \; dV = -\int_M u\Delta\, v_k \; dV + \int_{\partial M} u\partial_\nu v_k \; dA.$$

Now $\langle \nabla u, \nabla v_k\rangle_g$ converges to $\langle \nabla u, \nabla v\rangle_g$ in $L^1(M)$ and, since $s > 3/2$, $u\partial_\nu v_k$ converges to $u\partial_\nu v$ in $L^1(\partial M)$. If $s \geq 2$, then $u\Delta\, v_k$ converges to $u\Delta\, v$ in $L^1(M)$ as well, so this establishes integration by parts for $s \geq 2$. For the case $3/2 < s < 2$ we only need to be careful about the term

$$-\int_M u\Delta\, v_k \; dV.$$

On a compact manifold with boundary N, $H^\sigma(N) = \mathring{H}^\sigma(N)$ for $0 \leq \sigma < 1/2$ (the upper limit $1/2$ occurs since functions in H^σ with $\sigma > 1/2$ have trace values on the boundary). Using this fact in boundary charts shows that $H^\sigma_\delta(M) = \mathring{H}^\sigma_\delta(M)$ for any $\delta \in \mathbb{R}$ and $\sigma \in [0, 1/2)$. Let $\sigma = 2 - s$, so $0 \leq \sigma < 1/2$. Then

$$\Delta_{\,g} v_k \in H^{s-2}_{\delta-2} = H^{-\sigma}_{\delta-2},$$

and

$$u \in C^\infty_{\mathrm{c}}(M) \subset H^\sigma_{2-n-\delta}(M) = \mathring{H}^\sigma_{2-n-\delta}(M).$$

Hence,

$$\int_M u\Delta_{\,g} v_k \; dV_g = \langle \Delta_{\,g} v_k, u\rangle_{(M,g)}$$

where $\langle \cdot, \cdot\rangle_{(M,g)} : H^{-\sigma}_{\delta-2} \times \mathring{H}^\sigma_{2-n-\delta} \to \mathbb{R}$ is the duality pairing from Lemma 3.10. From the continuity of this bilinear form we find in the limit

$$\int_M \langle \nabla u, \nabla v\rangle_g \; dV = -\langle \Delta\, v, u\rangle_{(M,g)} + \int_{\partial M} u\partial_\nu v \; dA. \tag{4.27}$$

for all $v \in H^s_\delta(M)$ and $u \in C^\infty_c(M)$. From the continuity in $u \in H^1_{2-n-\delta}(M)$ of each term in (4.27) we obtain an integration by parts formula for $v \in H^s_\delta(M)$ and $u \in H^1_{2-n-\delta}(M)$. These arguments extend in the same way to the familiar integration by parts formula for the vector Laplacian, and we have the following.

Proposition 4.14 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. If $u \in H^1_{\delta_1}$ and $v \in H^\sigma_{\delta_2}$ with $\sigma > 3/2$ and $\delta_1 + \delta_2 \leq 2 - n$, then*

$$\int_M \langle \nabla u, \nabla v \rangle_g \; dV = -\langle \Delta\, v, u \rangle_{(M,g)} + \int_{\partial M} u \partial_\nu v \; dA.$$

Similarly, if $X \in H^1_{\delta_1}$ and $Y \in H^\sigma_{\delta_2}$ are vector fields we have

$$\int_M \langle \mathbb{L}X, \mathbb{L}Y \rangle_g \; dV = -\langle \Delta_{\mathbb{L}} Y, X \rangle_{(M,g)} + \int_{\partial M} B_{\mathbb{L}} Y(\nu, X) \; dA.$$

Our principal use of integration by parts is to show, for example, that the kernel of the Laplacian on acting on $H^s_\delta(\mathbb{R}^n)$ with $\delta < 0$ is trivial. We then have if $\Delta\, u = 0$ then

$$\int_M \langle \nabla u, \nabla u \rangle_g \; dV = -\langle \Delta\, u, u \rangle_{(M,g)}$$

and hence u is constant so long as we can justify that in fact $u \in H^s_{\delta'}$ for some $\delta' \leq \frac{2-n}{2}$. The following Lemma, analogous to Proposition 3.1 of [CO81], provides this justification.

Lemma 4.15 *Suppose A is an elliptic operator in $\mathcal{L}^{2,s}_\rho$ on $\mathbb{R}^n$ with $n \geq 3$, $s > n/2$ and $\rho < 0$. If $u \in H^s_\delta(\mathbb{R}^n)$ for some $\delta < 0$ satisfies Au is compactly supported, then $u \in H^s_{\delta'}(\mathbb{R}^n)$ for all $\delta' \in (2-n, 0)$.*

Proof: Let $A = A_\infty + R$ where A_∞ is the homogeneous constant coefficient linear elliptic operator giving the principal part of A at ∞. Then, since Au is compactly supported,

$$A_\infty u = Au - Ru \in H^{s-m}_{\delta-m+\rho}.$$

Since A is an isomorphism acting on $H^s_{\delta'}$ for each $\delta' \in (m-n, 0)$, we conclude that $u \in H^s_{\delta'}$ for each $\delta' \in (\max(m-n, \delta+\rho), 0)$. Iterating this argument yields the desired result. □

Chapter 5

TOOLS FOR THE CONFORMAL METHOD

In this chapter we establish three main tools for working with the constraint equations in our applications. First, we show that we can always solve

$$\Delta_{\mathbb{L}_g} W = -\mathrm{div}_g\, S$$
$$B_{\mathbb{L}} W = \omega \quad \text{on } \partial M$$

to obtain a transverse-traceless tensor $\sigma = \mathbb{L}W + S \in H^{s-1}_{\delta-1}$ with desired properties on the boundary. Second, we prove an existence theorem for the semilinear equation

$$\begin{aligned} -\Delta\, u &= F(x,u) \\ \partial_\nu u &= f(x,u). \quad \text{on } \partial M \end{aligned} \tag{5.1}$$

This will be our main tool for solving the Lichnerowicz equation. Finally, we establish some properties of metrics with $\lambda_g > 0$ that allow us to simplify our analysis of the Lichnerowicz equation.

A remark about manifold dimension is in order. Although we only solve the Lichnerowicz equation on 3-manifolds, our results for the vector Laplacian and the conformal invariant λ_g are of independent interest on general n-manifolds. This motivates us to work with $n \geq 3$ where possible (the case $n = 2$ is special and we do not treat it). For convenience, however, we restrict to $n = 3$ in the existence theorem for equation (5.1); see also the remark following Proposition 5.9.

5.1 The Vector Laplacian

Let $\mathcal{P}_{\mathbb{L}}$ be the operator

$$\mathcal{P}_{\mathbb{L}} : H^s_\delta(M) \to H^{s-2}_{\delta-2} \oplus H^{s-\frac{3}{2}}(\partial M)$$
$$X \mapsto (\Delta_{\mathbb{L}} X, B_{\mathbb{L}} X).$$

We wish to show that $\mathcal{P}_{\mathbb{L}}$ is an isomorphism acting on a suitable range of Sobolev spaces.

Proposition 5.1 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. If $\delta \in (2-n, 0)$ then $\mathcal{P}_{\mathbb{L}}$ is Fredholm with index 0. Moreover, the kernel of $\mathcal{P}_{\mathbb{L}}$ is the set of conformal Killing fields in $H^s_\delta(M)$.*

Proof: The a-priori estimate from Proposition 4.13 shows that $\mathcal{P}_{\mathbb{L}}$ is semi-Fredholm. To show it is Fredholm, we consider a sequence of smooth metrics $\{g_k\}_{k=1}^\infty$ such that $g - g_k \to 0$ in $H^s_\rho(M)$. The corresponding operators $\mathcal{P}_{\mathbb{L}^k}$ are well known to be Fredholm with index 0 and they converge as operators to $\mathcal{P}_{\mathbb{L}}$. The index of a semi-Fredholm operator is locally constant [Sc02] (that is, if L is semi-Fredholm there exists a neighbourhood L in which every operator is semi-Fredholm and has the same index as L). Hence the index of $\mathcal{P}_{\mathbb{L}}$ is 0. In particular, it is an isomorphism if and only if it has trivial kernel. On the other hand, if $X \in \ker(\Delta_{\mathbb{L}}, B_{\mathbb{L}})$ then Lemma 4.15 implies that $X \in H^s_{\delta'}$ for every $\delta' \in (2-n, 0)$. In particular, we can pick $\delta' \leq (2-n)/2$ and therefore integrate by parts to obtain

$$0 = \langle -\Delta_{\mathbb{L}} X, X \rangle_g = \int_M |\mathbb{L}X|^2 \; dV - \int_{\partial M} \mathbb{L}X(\nu, X) \, dA = \int_M |\mathbb{L}X|^2 \; dV.$$

So X is a conformal Killing field. $\square$

Proposition 5.1 shows that $\mathcal{P}_{\mathbb{L}}$ is an isomorphism if and only if (M, g) has no conformal Killing fields vanishing at infinity. This fact is well known for classical metrics. It was proved in [CO81] for C^2 metrics and in particular for metrics of class H^s_ρ where $s > n/2 + 2$. The level of regularity required was reduced in [Ma03] to H^k_ρ where k is an integer and $k > n/2 + 1$. Moreover, it was shown by means of a rescaling argument that if $k > n/2$, then any conformal Killing vanishing at infinity must vanish identically outside a compact set. More recently, Bartnik has shown, as a special case of [Ba04] Theorem 3.6, that when $n = 3$ there are no conformal Killing fields for metrics of class H^2_ρ. We now prove, augmenting the rescaling technique of [Ma03], that these results can be extended to metrics of class H^s_ρ with $s > n/2$.

Although the method of proof is quite different from that in [CO81] and its generalization in [Ba04], the spirit is the same. First, we show that if X is a conformal Killing field, then it vanishes outside a compact set. Second, we show that the zero set extends to include the whole manifold. Both cases rely on a rescaling argument to construct a conformal Killing field for the Euclidean metric. The fact that the Euclidean metric has no conformal Killing fields vanishing at infinity will imply that X vanishes outside a compact set, and the fact that the Euclidean metric has no conformal Killing fields Y such that $Y(0) = 0$ and $\nabla Y(0) = 0$ will allow us to see that the 0 set of X extends to all of M.

To make these arguments work, we need to know what happens when we rescale a metric in from infinity or out from a point x_0. The following Lemmas show that with suitable smoothness and decay hypotheses we obtain in the limit the value of the metric at infinity or at x_0, respectively.

Lemma 5.2 *Suppose $s \geq 0$ and $\delta \in \mathbb{R}$, and consider the family of operators $F_r : H^s_\delta(\mathbb{R}^n) \to H^s_\delta(E_1)$ given by $F_r u = \mathcal{S}_r u|_{E_1}$. Then for $r \geq 1$*

$$||F_r||_{H^s_\delta(\mathbb{R}^n)} \lesssim r^\delta.$$

In particular, if $\delta < 0$ and $u \in H^s_\delta$, then $F_r u$ converges to 0 in $H^s_\delta(E_1)$.

Proof: It is enough to consider the case s is an integer, since

$$||F_r||_{H^s_\delta} \lesssim \left(||F_r||_{H^{[s]}_\delta}\right)^{1-(s-[s])} \left(||F_r||_{H^{[s]+1}_\delta}\right)^{s-[s]}.$$

Let χ be a cutoff function equal to 1 on E_1 and vanishing on $B_{\frac{1}{2}}$, and define $M_\chi : H^s_\delta(\mathbb{R}^n) \to H^s_\delta(\mathbb{R}^n)$ by $M_\chi u = \chi u$. Then $F_r = R \circ M_\chi \circ \mathcal{S}_r$ where $R : H^s_\delta(\mathbb{R}^n) \to H^s_\delta(E_1)$ is the restriction map. Since R is continuous, it is enough to show

$$||M_\chi \circ \mathcal{S}_r||_{H^s_\delta(\mathbb{R}^n)} \lesssim r^\delta.$$

Using the equivalent norm $||\cdot||_{W^{k,2}_\delta}$ on H^k_δ when $k \geq 0$ is an integer we have

$$\begin{aligned} ||M_\chi \mathcal{S}_r u||^2_{H^k_\delta(E_1)} &= \sum_{|\alpha|\leq k} \int_{E_1} \langle x \rangle^{-2\delta-n+|\alpha|} |\partial_\alpha \mathcal{S}_r u|^2 \; dV + \\ &\qquad + \sum_{|\alpha|\leq k} \int_{A_1} \langle x \rangle^{-2\delta-n+|\alpha|} |\partial_\alpha \chi \mathcal{S}_r u|^2 \; dV \\ &\lesssim \sum_{|\alpha|\leq k} \int_{E_{\frac{1}{2}}} \langle x \rangle^{-2\delta-n+|\alpha|} |\partial_\alpha \mathcal{S}_r u|^2 \; dV \\ &= \sum_{|\alpha|\leq k} \int_{E_{\frac{1}{2}}} \langle x \rangle^{-2\delta-n+2|\alpha|} r^{|\alpha|} |\mathcal{S}_r \partial_\alpha u|^2 \; dV \\ &= \sum_{|\alpha|\leq k} r^{2\delta} \int_{E_{\frac{r}{2}}} \langle x/r \rangle^{-2\delta-n+2|\alpha|} r^{-2\delta-n+|\alpha|} |\partial_\alpha u|^2 \; dV. \end{aligned}$$

But for fixed a and any $x \geq r/2 \geq 1/2$,

$$\langle x/r \rangle^a \, r^a \lesssim \langle x \rangle^a \, .$$

Hence

$$||M_\chi \mathcal{S}_r u||_{H^k_\delta(\mathbb{R}^n)} \lesssim r^\delta ||u||_{H^k_\delta(\mathbb{R}^n)}$$

as claimed. □

In fact, the previous lemma can also be proved without the restriction $s \geq 0$; all that is required is the decay condition $\delta < 0$ to obtain a limit of 0 upon successive rescaling. By contrast, when we blow up about a point there is no restriction on the decay, but there is a restriction on the regularity. We require $s > n/2$.

Lemma 5.3 *Suppose $s > n/2$ and suppose $\beta \in (0,1)$ satisfies $\beta \leq s - n/2$. Consider the family of operators $G_r : H^s(B_1(\mathbb{R}^n)) \to H^s(B_1(\mathbb{R}^n))$ given by $G_r u(x) = u(rx) - u(0)$ for $r \in (0,1)$. Then for every $r \in (0,1)$,*

$$||G_r||_{H^s(B_1)} \lesssim r^\beta.$$

Proof: Let $\theta = s - [s]$ and let

$$\frac{1}{q} = \frac{1}{2} + \frac{1-\theta}{n}$$
$$\frac{1}{p} = \frac{1}{2} - \frac{\theta}{n}.$$

Then $H^s(B_1) = [W^{[s],p}(B_1), W^{[s]+1,q}(B_1)]_\theta$. Noting that

$$[s] - \frac{n}{p} = [s] + 1 - \frac{n}{q} = s - \frac{n}{2},$$

from interpolation it is enough to prove that if $k > n/p$ and if $\beta \in (0,1)$ satisfies $\beta \leq k - n/p$, then

$$||G_r||_{W^{k,p}(B_1)} \lesssim r^\beta$$

for $r \in (0,1)$.

We will use the equivalent norm on $W^{k,p}(B_1)$

$$||u||^p_{\tilde{W}^{k,p}(B_1)} = ||u||^p_{L^p(B_1)} + \sum_{|\alpha|=k} ||\partial_\alpha u||^p_{L^p(B_1)}.$$

If $n/p+1 > k > n/p$, then $\beta = k - n/p$ and u is Hölder continuous with exponent β. So

$$\sup_{B_r(0)} |u(x) - u(0)| \lesssim ||u||_{\tilde{W}^{k,p}(B_1)} r^{\beta}. \tag{5.2}$$

On the other hand, if $k > n/p+1$, then u is Hölder continuous to every order and the estimate (5.2) also holds. Hence

$$\begin{aligned} ||G_r u||^p_{L^p(B_1)} &= \int_{B_1} |u(rx) - u(0)|^p \; dV \\ &= r^{-n} \int_{B_r} |u(x) - u(0)|^p \; dV \\ &\lesssim ||u||_{\tilde{W}^{k,p}(B_1)} r^{p\beta}. \end{aligned} \tag{5.3}$$

On the other hand, if $|\alpha| = k$ then

$$\begin{aligned} ||\partial_\alpha G_r u||^p_{L^p(B_1)} &= \int_{B_1} r^{kp} |(\partial_\alpha u)(rx)|^p \; dV \\ &= r^{kp-n} \int_{B_r} |(\partial_\alpha u)|^p \; dV \\ &\leq r^{p\beta} \int_{B_r} |(\partial_\alpha u)|^p \; dV. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4) we obtain

$$||G_r u||_{\tilde{W}^{k,p}(B_1)} \lesssim r^{\beta} ||u||_{\tilde{W}^{k,p}(B_1)}$$

as claimed. □

With Lemmas 5.2 and 5.3 in hand, we show using the a priori estimates of Chapter 4 that a conformal Killing field decaying at infinity vanishes in a neighbourhood of infinity.

Lemma 5.4 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. If $X \in H^s_\delta$ with $s > n/2$ and $\delta < 0$ is a conformal Killing field, then it vanishes outside some compact set.*

Proof: We assume for simplicity that M has a single end. Working in end coordinates, we define a sequence of metrics $\{g_k\}_{k=1}^{\infty}$ on the exterior region E_1 via $g_k(x) = g(2^k x)$. Since $\rho < 0$, it follows from Lemma 5.2 that $g_k - \overline{g}$ converges to 0 in $H^s_\rho(E_1)$. Hence the associated operators $\Delta^k_{\mathbb{L}}$, $\mathbb{L}^k$, and B^k converge to their Euclidean analogues as operators on $H^s_\delta(E_1)$.

Suppose, to produce a contradiction, that X is not identically 0 outside any exterior region E_R. Let $\hat{X}_k(x) = X(2^k x)$ and let $X_k = \hat{X}_k / ||\hat{X}_k||_{H^s_\delta(E_1)}$. Since the sequence $\{X_k\}_{k=0}^{\infty}$ is bounded in

H^s_δ, we conclude after reducing to a subsequence that the sequence converges strongly in $H^{s-1}_{\delta'}$ for any $\delta' \in (\delta, 0)$ to some vector field X_0.

We can assume without loss of generality that $\delta \in (2-n, 0)$. So from Proposition 4.13, the H^s_δ boundedness of the sequence $\{X_k\}_{k=1}^\infty$, and the identities $\mathbb{L}^k X_k = 0$, it follows that

$$\begin{aligned}||X_{k_1} - X_{k_2}||_{H^s_\delta(E_1)} \lesssim ||\overline{\Delta}_{\mathbb{L}} - \Delta^{k_1}_{\mathbb{L}}||_{H^s_\delta(E_1)} + ||\overline{\Delta}_{\mathbb{L}} - \Delta^{k_2}_{\mathbb{L}}||_{H^s_\delta(E_1)} + ||\overline{B} - B^{k_1}||_{H^s_\delta(E_1)} \\ + ||\overline{B} - B^{k_2}||_{H^s_\delta(E_1)} + ||X_{k_1} - X_{k_2}||_{H^{s-1}_{\delta'}(E_1)}.\end{aligned}$$

We conclude $\{X_k\}_{k=1}^\infty$ is Cauchy in $H^s_\delta(E_1)$ and hence converges in H^s_δ to X_0. Since $||X_k|| = 1$, X_0 cannot be identically zero. Moreover, since X_k is a conformal Killing field for g_k, it follows that X_0 is a conformal Killing field for $\overline{g}$. But $\overline{g}$ does not admit any nontrivial conformal Killing fields in H^s_δ, a contradiction. □

Let U be the interior of the zero set of a conformal Killing field X. From Lemma 5.4 we know that U is non-empty. We want to blow up the vector field about a point $x_0 \in \partial U$ to obtain a conformal Killing field Y for the Euclidean metric that satisfies $Y(0) = 0$ and $\nabla Y(0) = 0$. To do this, we must choose the point x_0 carefully. One could imagine, for example, that if x_0 were a cusp point for ∂U, then we would find $Y(0) = 0$ in the limit but we would have no control on ∇Y. Fortunately, we can always find a suitable centre for rescaling.

Proposition 5.5 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. If $X \in H^s_\delta$ with $s > n/2$ and $\delta < 0$ is a conformal Killing field, then it vanishes identically.*

Proof: As in the discussion above, let U be the interior of $X^{-1}(0)$, so $U \neq \emptyset$. To show that $U = \operatorname{int} M$, it is enough to show that it has empty boundary. Suppose, to produce a contradiction, that x_0 is a boundary point of U. Working in local coordinates about x_0, we can assume $M = \mathbb{R}^n$. Hereafter, all balls and distances are computed with respect to the flat background metric. Let $y_0 \in U$ and let $r = d(y_0, \partial U)$. Then $B_r(y_0) \subset U$, and there exists some point $z_0 \in B_r(y_0) \cap \partial U$. After making an affine change of coordinates, we can assume $z_0 = 0$ and $g(0) = \overline{g}$.

We construct a sequence of metrics $\{g_k\}_{k=1}^\infty$ on the unit ball by taking $g_k(x) = g(2^{-k}x)$. Since $s > n/2$ and $g(0) = \overline{g}$, it follows from Lemma 5.3 that $g_k - \overline{g}$ converges to 0 in $H^s(B_1)$. It follows that the associated maps $\Delta^k_{\mathbb{L}}$, $\mathbb{L}^k$, and B^k converge to their Euclidean counterparts as operators on $H^s(B_1)$.

We construct vector fields X_k on B_1 by setting $\hat{X}_k(x) = X(2^{-k}x)$ and letting

$$X_k = \hat{X}_k / ||\hat{X}_k||_{H^s(B_1)}$$

This normalization is possible since X is not identically 0 on $B_{2^{-k}}$. Since the sequence is bounded in H^s, we conclude that the sequence converges strongly in $H^{s-1}(B_1)$ to some $X_0 \in H^s(B_1)$. Moreover, from the choice of the point z, it follows that X_k vanishes on an open cone K independent of k.

Arguing as in Lemma 5.4, replacing the use of Proposition 4.13 with Proposition 4.12, we conclude X_0 is a conformal Killing field for $\overline{g}$ and X_k converges in $H^s(B_1)$ to X_0. In particular, X_0 is a nontrivial conformal Killing field for $\overline{g}$ on B_1 that vanishes on an open cone. But any such conformal Killing field must vanish identically, a contradiction. □

Combining Propositions 5.1 and 5.5 we immediately obtain

Theorem 5.6 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. If $\delta \in (2-n, 0)$ then $\mathcal{P}_{\mathbb{L}} : H^s_\delta \to H^{s-2}_{\delta-2} \times H^{s-\frac{3}{2}}(\partial M)$ is an isomorphism.*

5.2 The Method of Sub- and Supersolutions

Our existence theorem for solutions of the Lichnerowicz equation relies on the well-known method of sub and super-solutions. Versions of this technique have been used before to find solutions of the Lichnerowicz equation for regular metrics, e.g. using the Leray-Schauder fixed point theorem in [CB72] or via a constructive approach in [Is95]. The constructive method has subsequently been extended to weaker classes of metrics [CB03][Ma03]. In this section we provide a version of the barrier construction that accommodates both semilinear boundary conditions and rough metrics.

Consider the boundary value problem

$$\begin{aligned} -\Delta\, u &= F(x,u) \\ \partial_\nu u &= f(x,u) \quad \text{on } \partial M \end{aligned} \tag{5.5}$$

on an asymptotically Euclidean manifold. A subsolution of equation (5.5) is a function u_- that satisfies

$$\begin{aligned} -\Delta\, u_- &\le F(x,u_-) \\ \partial_\nu u_- &\le f(x,u_-) \quad \text{on } \partial M \end{aligned}$$

and a supersolution u_+ is defined similarly with the inequalities reversed. In Proposition 5.9 below, we show that if there exists a subsolution u_- and a supersolution u_+ decaying at infinity and satisfying $u_- \le u_+$, then there exists a solution u satisfying $u_- \le u \le u_+$.

The proof of Proposition 5.9 relies on properties of the associated linearized operator

$$\begin{aligned} -\Delta u + Vu &= F \\ \partial_\nu u + \mu u &= f, \quad \text{on } \partial M \end{aligned} \tag{5.6}$$

where V, μ, F, and f are functions of x alone. We write $\mathcal{P}_{V,\mu}$ for the operator $(-\Delta + V,\, \partial_\nu + \mu \mid_{\partial M})$ mapping $H^s_\delta(M)$ to $H^{s-2}_{\delta-2}(M) \times H^{s-\frac{3}{2}}(\partial M)$.

Proposition 5.7 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. Suppose also that $V \in H^{s-2}_{\rho-2}(M)$ and $\mu \in H^{s-\frac{3}{2}}(\partial M)$. Then for $\delta \in (2-n, 0)$, $\mathcal{P}_{V,\mu}$ mapping $H^s_\delta(M)$ to $H^{s-2}_{\delta-2}(M) \times H^{s-\frac{3}{2}}(\partial M)$ is Fredholm with index 0. Moreover, if $V \ge 0$ and $\mu \ge 0$, then $\mathcal{P}_{V,\mu}$ is an isomorphism.*

Proof: When $V = 0$ and $\mu = 0$, the proof that $\mathcal{P}_{0,0}$ is Fredholm with index 0 proceeds exactly as in Proposition 5.1. On the other hand, $\mathcal{P}_{V,\mu}$ is a compact perturbation of $\mathcal{P}_{0,0}$ and hence also has index 0. The follow maximum principle proves that when $V \ge 0$ and $\mu \ge 0$, then $\ker \mathcal{P}_{V,\mu} = 0$ and hence $\mathcal{P}_{V,\mu}$ is an isomorphism. $\square$

We recall that if $V \in H^{s-2}_{\delta-2}$, we say that $V \ge 0$ if $\langle V, \phi \rangle_{(g,M)} \ge 0$ for every non-negative, smooth, compactly supported function ϕ. From a density argument, this is equivalent to the same condition holding for $\phi \in H^{2-s}_{2-n-\delta}$. With this definition in mind, we have the following weak maximum principle.

Lemma 5.8 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$. Suppose also that $V \in H^{s-2}_{\rho-2}(M)$, $\mu \in H^{s-\frac{3}{2}}(\partial M)$, and $V \ge 0$ and $\mu \ge 0$. If $u \in H^s_{\mathrm{loc}}$ satisfies*

$$\begin{aligned} -\Delta u + Vu &\le 0 \\ \partial_\nu u + \mu u &\le 0 \end{aligned} \tag{5.7}$$

and if $u^{(+)} = \max(u, 0)$ is $o(1)$ on each end of M, then $u \le 0$. In particular, if $u \in H^s_\delta(M)$ for some $\delta < 0$ and u satisfies (5.7)*, then $u \le 0$.*

Proof: Fix $\epsilon > 0$, and let $v = (u-\epsilon)^{(+)}$. Since $u^{(+)} = o(1)$ on each end, we see v is compactly supported. Since $u \in H^1_{\text{loc}}$, $v \in H^1_{\text{loc}}$ also. Since $u \in H^s_{\text{loc}}$ with $s > n/2$, and since v is compactly supported, we conclude $uv \in H^1(M)$ and $uv \geq 0$. Since $V \in H^{s-2}_{\text{loc}}(M)$, and since $s-2 \geq -1$ we can apply V to uv. Since u satisfies (5.7) we obtain

$$-\left\langle \Delta u, v\right\rangle_{(M,g)} \leq \left\langle V, uv\right\rangle \leq 0.$$

Integrating by parts and using (5.7) again we find

$$\begin{aligned}\int_M \left\langle \nabla u, \nabla v\right\rangle_g \, dV &\leq \int_{\partial M} \partial_\nu uv \, dA \\ &\leq -\int_{\partial M} \mu uv \, dA \\ &\leq 0.\end{aligned}$$

Now $\nabla v = \nabla(u-\epsilon)^{(+)} = \nabla u$ wherever $u > \epsilon$, and ∇v vanishes otherwise (see, e.g. [GT99] Lemma 7.6). Hence $\left\langle \nabla u, \nabla v\right\rangle_g = \left\langle \nabla v, \nabla v\right\rangle_g$. We conclude

$$\int_M |\nabla v|^2 \, dV \leq 0.$$

So v is constant and compactly supported. We conclude $u \leq \epsilon$ on M, and taking ϵ to 0 proves $u \leq 0$. □

We now turn to the existence proof for the nonlinear problem

$$\begin{aligned}-\Delta\, u &= F(x,u) \\ \partial_\nu u &= f(x,u) \quad \text{on } \partial M.\end{aligned} \tag{5.8}$$

We assume for simplicity that the nonlinearities F and f have the form

$$\begin{aligned}F(x,y) &= \sum_{j=1}^{l} F_j(x) G_j(y) \\ f(x,y) &= \sum_{j=1}^{m} f_j(x) g_j(y).\end{aligned}$$

Proposition 5.9 *Suppose*

1. *(M^3, g) is AE of class H^s_ρ with $s > 3/2$ and $\rho < 0$,*

2. $u_-,\ u_+ \in H^s_\delta$ *with and* $\delta \in (2-n, 0)$ *are a subsolution and a supersolution respectively of* (5.8) *such that* $u_- \leq u_+$,

3. *each* $F_j \in H^{s-2}_{\delta-2}(M)$ *and* $f_j \in W^{s-\frac{3}{2},(}(\partial M)$ *are non-negative,*

4. *each* G_j *and* g_j *are smooth on* $I = [\inf(u_-), \sup(u_+)]$.

Then there exists a solution u *of* (5.8) *such that* $u_- \leq u \leq u_+$.

Proof: We first assume $3/2 < s \leq 2$. Let

$$V(x) = \sum_{j=1}^{l} F_j(x) \left| \min_I G'_j \right|$$
$$\mu(x) = \sum_{j=1}^{m} f_j(x) \left| \min_I g'_j \right|,$$

so that $V \in H^{s-2}_{\delta-2}(M)$, $\mu \in H^{s-\frac{3}{2}}(\partial M)$, and both are nonnegative. Let $F_V(x,y) = F(x,y) + V(x)y$ and $f_\mu(x,y) = f(x,y) + \mu(x)y$ so that F_V and f_μ are both non-decreasing in y. By this we mean that if $u, v \in H^s_\delta(\Omega)$ satisfy $u \geq v$, then $F_V(x,u) - F_V(x,v) \geq 0$ as a distribution. Let $L_V = -\Delta + V$, and let $B_\mu = (\partial_\nu + \mu)\,|_{\partial M}$. From Proposition 5.7 we have (L_V, B_μ) is an isomorphism acting on H^s_δ.

We construct a sequence of functions as follows. Let $u_0 = u_+$, and for $i \geq 1$ let u_i be the solution of

$$L_V\, u_{i+1} = F_V(x, u_i)$$
$$B_\mu\, u_{i+1} = f_\mu(x, u_i).$$

Now

$$L_V(u_1 - u_0) \leq F_V(x, u_0) - F(x, u_0) - V(x)u_0 = 0$$
$$B_\mu(u_1 - u_0) \leq f_\mu(x, u_0) - f(x, u_0) - \mu(x)u_0 = 0$$

since u_0 is a super-solution. Moreover,

$$L_V(u_1 - u_-) \geq F_V(x, u_+) - FV(x, u_-) \geq 0$$
$$B_\mu(u_1 - u_-) \geq f_\mu(x, u_+) - f_\mu(x, u_-) \geq 0$$

since $u_+ \geq u_-$. From Lemma 5.8 we conclude $u_0 \geq u_1 \geq u_-$. Now suppose $u_0 \geq u_1 \geq \cdots \geq u_i \geq u_-$. Then

$$L_V(u_{i+1} - u_i) = F_V(x, u_i) - F(x, u_{i-1}) \leq 0$$

$$B_\mu(u_{i+1} - u_i) = f_\mu(x, u_i) - f(x, u_{i-1}) \leq 0$$

Hence $u_{i+1} \leq u_i$. Also,

$$L_V(u_{i+1} - u_-) = F_V(x, u_i) - F(x, u_-) \geq 0$$

$$B_\mu(u_{i+1} - u_-) = f_\mu(x, u_i) - f(x, u_-) \geq 0.$$

So $u_i \geq u_{i+1} \geq u_-$. We obtain by induction for the entire sequence $u_+ = u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_-$.

We claim the sequence $\{u_i\}_{i=1}^{\infty}$ is bounded in $H^s_\delta(M)$. From Proposition 5.7 we can estimate

$$||u_{i+1}||_{H^s(M)} \lesssim ||F_V(x, u_i)||_{H^{s-2}(M)} + ||f_\mu(x, u_i)||_{H^{s-\frac{3}{2}}}. \tag{5.9}$$

We turn to estimating each term on the right-hand side of (5.9). $F_V(x, u_i)$ is a sum of terms of the form

$$F(x)G(u_i)$$

where $F \in H^{s-2}_{\delta-2}(M)$. From Lemma 5.10 proved below we have

$$||F(x)G(u_i)||_{H^{s-2}_{\delta-2}} \lesssim ||F||_{H^{s-2}_{\delta-2}} \left[||G(u_i)||_{L^\infty} + ||G'(u_i)||_{L^\infty} ||u||_{H^{s'}_\delta} \right]$$

for fixed $s' \in (n/2, s)$. Since $u_- \leq u_i \leq u_+$, we have uniform estimates for each of the terms $||G(u_i)||_{L^\infty}$ and $||G'(u_i)||_{L^\infty}$. Hence

$$||F_V(x, u_i)||_{H^{s-2}_{\delta-2}(M)} \lesssim 1 + ||u||_{H^{s'}_\delta(M)}. \tag{5.10}$$

Turning to estimates for $f_\mu(x, u_i)$ we have a sum of terms of the form

$$f(x)g(u_i)$$

where $f \in H^{s-\frac{3}{2}}(\partial M)$. Since $s - \frac{3}{2} \in [-1, 1]$ we find from Lemma 5.11 below that

$$||f(x)g(u_i)||_{H^{s-\frac{3}{2}}(\partial M)} \lesssim ||f||_{H^{s-\frac{3}{2}}(\partial M)} \left[||g(u_i)||_{L^\infty(\partial M)} + ||g'(u_i)||_{L^\infty(\partial M)} ||u||_{H^{s'-\frac{1}{2}}(\partial M)} \right].$$

Again we have uniform estimates for $||g(u_i)||_{L^\infty(\partial M)}$ and $||g'(u_i)||_{L^\infty(\partial M)}$. From the trace theorem we have

$$||u||_{H^{s'-\frac{1}{2}}(\partial M)} \lesssim ||u||_{H^{s'}_\delta(M)}$$

and therefore

$$||f_\mu(x,u_i)||_{H^{s-\frac{3}{2}}} \lesssim 1 + ||u_i||_{H^{s'}_\delta(M)}. \tag{5.11}$$

Now from interpolation we know that for any $\epsilon > 0$ there is a constant $C(\epsilon)$ such that

$$||u||_{H^{s'}_\delta} \leq C(\epsilon)||u||_{H^0_\delta} + \epsilon||u||_{H^s_\delta} \tag{5.12}$$

Again, since $u_- \leq u_i \leq u_+$, we have uniform estimates for $||u||_{H^0_\delta}$. Combining (5.9), (5.10), (5.11) and (5.12) we obtain, taking ϵ sufficiently small,

$$||u_{i+1}||_{H^s_\delta(M)} \leq \frac{1}{2}||u_i||_{H^s_\delta(M)} + C$$

for some constant C independent of i. Iterating this inequality we obtain a bound for all i

$$||u_i||_{H^s_\delta(M)} \leq ||u_+||_{H^s_\delta(M)} + 2C.$$

Hence some subsequence of $\{u_i\}_{i=1}^\infty$ (and by monotonicity, the whole sequence) converges weakly in $H^s_\delta(M)$ to a limit u_∞.

It remains to see u_∞ is a solution of (5.8). Now u_i converges strongly to u_∞ in $H^{s'}_{\delta'}$ for any $s' < s$ and $\delta' > \delta$, and also converges uniformly on compact sets. Hence for any $\phi \in C^\infty_c(M)$,

$$\begin{aligned}
\int_M (F_V(x,u_i) - V(x)u_{i+1})\,\phi\,dV &\to \int_M F(x,u_\infty)\phi\,dV \\
\int_{\partial M} (f_\mu(x,u_i) - \mu(x)u_{i+1})\,\phi\,dA &\to \int_{\partial M} f(x,u_\infty)\phi\,dA \\
\int_M \langle\nabla u_{i+1},\nabla\phi\rangle\,\phi\,dV &\to \int_M \langle\nabla u_\infty,\nabla\phi\rangle\;dV.
\end{aligned}$$

So

$$\int_M \langle\nabla u_\infty,\nabla\phi\rangle\;dV = \int_M F(x,u_\infty)\phi\,dV + \int_{\partial M} f(x,u_\infty)\phi\,dA,$$

and an application of integration by parts shows u_∞ solves the boundary value problem.

To handle the case $s > 2$ we use a bootstrap. First suppose $4 \geq s \geq 2$. From the above we have a solution u in H^2_δ. Since $2 > n/2 = 3/2$ and since $2 > s-2 \in [0,2]$, we know from Lemma 3.6 and the remark following it that $c(x)f(u) \in H^{s-2}_{\delta-2}$. Since $-\Delta\,u \in H^{s-2}_{\delta-2}$, Proposition 5.7 implies $u \in H^s_\delta$. We obtain the result for all $s > 3/2$ by induction. $\square$

Remark 2 *The restriction $n = 3$ in Proposition 5.9 is a consequence of our choice to work with the spaces H^s_δ. It arises since H^2_δ is an algebra for $n = 3$, but not for $n > 3$. This is not a fundamental restriction. We showed in [Ma03] that a similar proposition can be proved for the Sobolev spaces $W^{k,p}_\delta$ where $k \geq 2$ and $k > n/p$. Every AE metric of class H^s_δ with $s > n/2$ and $n > 3$ is also of class $W^{[s],p}_\delta$ where $[s] > 2$ and where p satisfies $[s] > n/p$. So we can find a solution in $W^{[s],p}_\delta$, and in fact we can bootstrap from here to H^s_δ. For details, see [Ma04]. Since we have chosen to present here the linear theory the spaces H^s_δ only, rather than that for the spaces $W^{k,p}_\delta$, we restrict our attention to 3-manifolds.*

The following two lemmas complete the proof of Proposition 5.9.

Lemma 5.10 *Suppose (M^n, g) is an AE manifold of class H^s_ρ with $s > n/2$ and $\rho < 0$. Suppose F is a smooth function, $\delta \in \mathbb{R}$, $\sigma \in [-1, 1]$, and $s' \in (n/2, s)$. Then for every $u \in H^s_\rho$ and $v \in H^\sigma_\delta$ we have*

$$||vf(u)||_{H^\sigma_\delta(M)} \lesssim ||v||_{H^\sigma_\delta} \left[||f(u)||_{L^\infty}(M) + ||f'(u)||_{L^\infty}(M)||u||_{H^{s'}_\rho(M)} \right].$$

Proof: If $\sigma = 1$, then

$$\begin{aligned} ||vf(u)||_{H^1_\delta(M)} &\lesssim ||vf(u)||_{L^2_\delta(M)} + ||\nabla(vf(u))||_{L^2_{\delta-1}(M)} \\ &\lesssim ||vf(u)||_{L^2_\delta(M)} + ||\nabla v f(u)||_{L^2_{\delta-1}(M)} + ||vf'(u)\nabla u||_{L^2_{\delta-1}(M)} \\ &\lesssim ||v||_{H^1_\delta(M)} ||f(u)||_{L^\infty(M)} + ||vf'(u)\nabla u||_{L^2_{\delta-1}(M)}. \end{aligned}$$

To estimate the last term we note that since $v \in H^1_\delta$, it is also in L^p_δ where

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{n}.$$

On the other hand, $\nabla u \in L^q_{\rho-1}$ where

$$\frac{1}{q} = \frac{1}{2} - \frac{s'-1}{n}.$$

Since

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{2} - \frac{1}{n} + \frac{1}{2} - \frac{s'-1}{n} \\ &= \frac{1}{2} + \left(\frac{1}{2} - \frac{s'}{n}\right) \\ &< \frac{1}{2} \end{aligned}$$

we conclude

$$||vf'(u)\nabla u||_{L^2_{\delta-1}(M)} \lesssim ||v||_{H^1_\delta} ||f'(u)||_{L^\infty(M)} ||u||_{H^{s'}_\rho}. \tag{5.13}$$

This proves the result in the case $s = 1$.

We prove the case $s = -1$ by duality. If $w \in H^1_{-n-\delta}(M)$, then

$$\begin{aligned} \left| \langle vf(u), w \rangle_{(M,g)} \right| &\lesssim ||v||_{H^{-1}_\delta} ||f(u)w||_{H^1_{-n-\delta}(M)} \\ &\lesssim ||v||_{H^{-1}_\delta} \left[||f(u)||_{L^\infty}(M) + ||f'(u)||_{L^\infty}(M) ||u||_{H^{s'}_\rho(M)} \right] ||w||_{H^1_{-n-\delta}(M)}. \end{aligned}$$

Hence

$$||vf(u)||_{H^{-s}_\delta} \lesssim ||v||_{H^{-1}_\delta} \left[||f(u)||_{L^\infty}(M) + ||f'(u)||_{L^\infty}(M) ||u||_{H^{s'}_\rho(M)} \right].$$

We have therefore obtained the result for $s = -1$, and the result for all $s \in [-1, 1]$ now follows from interpolation. □

The corresponding fact for compact manifolds is proved identically. We omit the proof.

Lemma 5.11 *Suppose (M^n, g) is a compact Riemannian manifold of class H^s with $s > n/2$. Suppose F is a smooth function, $\sigma \in [-1, 1]$, and $\sigma' \in (n/2, s)$. Then for every $u \in H^s(M)$ and $v \in H^\sigma(M)$ we have*

$$||vf(u)||_{H^\sigma_\delta(M)} \lesssim ||v||_{H^\sigma_\delta} \left[||f(u)||_{L^\infty}(M) + ||f'(u)||_{L^\infty}(M) ||u||_{H^{s'}_\rho(M)} \right].$$

5.3 Conformal Transformations of Asymptotically Euclidean Manifolds

On a compact Riemannian manifold (M^n, g) without boundary, the Yamabe invariant λ_g is defined by

$$\lambda_g = \inf_{f \in C^\infty(M), f \not\equiv 0} \frac{\int_M a\, |\nabla f|^2 + Rf^2 \, dV}{||f||_{L^{2^*}}}$$

where $a = \frac{4(n-1)}{n-2}$ and where 2^* is the critical Sobolev exponent $\frac{2n}{n-2}$. There is a well known relationship between the Yamabe invariant, a geometric condition on (M, g), and an analytic condition on (M, g). Namely, the following are equivalent.

1. There is a metric $\tilde{g} \in [g]$ with scalar curvature everywhere positive (resp. negative, zero).

2. The Yamabe invariant λ_g is positive (resp. negative, zero).

3. The first non-zero eigenvalue of $-a\Delta_g + R$ is positive (resp. negative, zero).

In [Es92], Escobar extended the notion of Yamabe invariant to compact manifolds with boundary. Following this, [Ma03] made the natural extension of this definition to asymptotically Euclidean manifolds with boundary,

$$\lambda_g = \inf_{f \in C_c^\infty(M), f \not\equiv 0} \frac{\int_M a\,|\nabla f|^2 + Rf^2\, dV + \int_{\partial M} 2Hf^2\, dA}{||f||_{L^{2^*}}^2}.$$

For rough metrics, by $\int_M Rf^2\, dV$ we mean $\left\langle R, f^2 \right\rangle_{(M,g)}$. There will be no confusion on this point in practice, so we keep the more suggestive notation.

We want show that the condition $\lambda_g > 0$ is equivalent to a geometric condition and an analytic condition. Since the conformal Laplacian does not have eigenfunctions that vanish at infinity, the analytic condition cannot be expressed in terms of an eigenvalue. To state it, we consider instead the family of operators

$$\mathcal{P}_\eta = \left(-a\Delta + \eta R, \frac{a}{2}\partial_\nu + \eta H \Big|_{\partial M} \right).$$

When $\eta = 1$, This is related to the conformal change of scalar curvature and boundary mean curvature. If $\tilde{g} = \phi^{2\kappa} g$, where $\kappa = \frac{2}{n-2}$, then

$$\begin{aligned} R_{\tilde{g}} &= \phi^{-2\kappa-1} \left(-a\Delta_g \phi + R_g \phi \right) \\ H_{\tilde{g}} &= \phi^{-\kappa-1} \left(\frac{a}{2}\partial_{\nu_g}\phi + H_g \phi \right). \end{aligned}$$

Proposition 5.12 *Suppose (M^n, g) is AE of class H^s_δ with $s > n/2$, and $\delta \in (2-n, 0)$. Then the following conditions are equivalent:*

1. *There exists a conformal factor $\phi > 0$ such that $1 - \phi \in H^s_\delta(M)$ and such that $(M, \phi^{\frac{4}{n-2}} g)$ is scalar flat and ∂M is a minimal surface.*

2. *$\lambda_g > 0$.*

3. *For each $\eta \in [0,1]$, $\mathcal{P}_\eta$ is an isomorphism acting on $H^s_\delta(M)$.*

Proof: Suppose condition 1 holds. Since λ_g is a conformal invariant, we can assume that $R = 0$ and $H = 0$. By solving the equation

$$\begin{aligned} -a\Delta\, v &= \mathcal{R} \\ \frac{a}{2}\partial_\nu v &= 0 \end{aligned} \tag{5.14}$$

for some smooth positive $\mathcal{R} \in H^{s-2}_{\delta-2}$, we can make the conformal change corresponding to $\phi = 1+v$ to a metric with continuous positive scalar curvature R and a minimal surface boundary. Let K be the compact core of M. Since

$$||f||^2_{L^{2^*}} \lesssim ||\nabla f||^2_{L^2} + ||f||^2_{L^2(K)}, \tag{5.15}$$

we find

$$||f||^2_{L^{2^*}} \lesssim ||\nabla f||^2_{L^2} + ||R^{1/2} f||^2_{L^2}.$$

Hence $\lambda_g > 0$.

Now suppose condition 2 holds. To show condition 3 is true, it is enough to show each $\mathcal{P}_\eta$ has trivial kernel for each $\eta \in [0,1]$. When $\eta = 0$ the result is obvious, so we consider the case $\eta \in (0,1]$. Suppose, to produce a contradiction, that $\mathcal{P}_\eta u = 0$. From Lemma 4.15 we have $u \in H^s_{\delta'}$ for any $\delta' \in (2-n, 0)$. Fixing $\delta' \leq (2-n)/2$ we can integrate by parts to obtain

$$\begin{aligned} 0 = \int_M -au\Delta\, u + \eta R u^2 \; dV &= \int_M a\,|\nabla u|^2 + \eta R u^2 \; dV - \int_{\partial M} a \partial_\nu u u \; dA \\ &= \int_M a\,|\nabla u|^2 + \eta R u^2 \; dV + \eta \int_{\partial M} 2Hu^2 \; dA \\ &\geq \eta \left(\int_M a\,|\nabla u|^2 + Ru^2 \; dV + \int_{\partial M} 2Hu^2 \; dA \right) \end{aligned} \tag{5.16}$$

Since $\eta > 0$,

$$0 \geq \int_M a\,|\nabla u|^2 + Ru^2 \; dV + \int_{\partial M} 2Hu^2 \; dA.$$

Let u_k be a sequence of functions in $C^\infty_{\mathrm{c}}(M)$ converging in $H^s_{\delta'}(M)$ to u. The map taking u_k to

$$\int_M a\,|\nabla u_k|^2 + Ru_k^2 \; dV + \int_{\partial M} 2Hu_k^2 \; dA.$$

is continuous on $H^s_{\delta'}(M)$. Moreover, from (5.15) and the inequality $\delta' \leq (2-n)/2$ we have the continuous embeddings $H^s_{\delta'} \to H^1_{\delta'} \to L^{2^*}$. It follows that $||u_k||_{L^{2^*}}$ converges to $||u||_{L^{2^*}} \neq 0$. Hence

$$\begin{aligned} \lambda_g &\leq \lim_{k\to\infty} \frac{\int_M a\,|\nabla u_k|^2 + Ru_k^2 \; dV + \int_{\partial M} 2Hu_k^2 \; dA}{||u_k||^2_{L^{2^*}}} \\ &= \frac{\int_M a\,|\nabla u|^2 + Ru^2 \; dV + \int_{\partial M} 2Hu^2 \; dA}{||u||^2_{L^{2^*}}} \\ &\leq 0. \end{aligned}$$

But $\lambda_g > 0$ by hypothesis, so we have a contradiction.

Finally, suppose condition 3 holds. For each $\eta \in [0,1]$, let v_η be the unique solution in H^s_δ of

$$\mathcal{P}_\eta v_\eta = -\eta(R, H).$$

Letting $\phi_\eta = 1 + v_\eta$ we see

$$\begin{aligned} -a\Delta\,\phi_\eta + \eta R\phi_\eta &= 0 \\ \frac{a}{2}\partial_\nu\phi_\eta + \eta H\phi_\eta &= 0. \end{aligned} \tag{5.17}$$

To show $\phi_\eta > 0$ for all $\eta \in [0,1]$, we use a continuity argument. Let $I = \{\eta \in [0,1] : \phi_\eta > 0\}$. Since $v_0 = 0$, we have I is nonempty. Moreover, the set $\{v \in C^0_\delta : v > -1\}$ is open in C^0_δ. Since the map taking η to $v_\eta \in C^0_\delta$ is continuous, I is open. It suffices to show that I is closed. Suppose $\eta_0 \in \bar{I}$. Then $\phi_{\eta_0} \geq 0$. Since ϕ_η solves (5.17), and since ϕ_{η_0} tends to 1 at infinity, Lemma 5.14 proved below implies $\phi_{\eta_0} > 0$. Hence $\eta_0 \in I$ and I is closed.

Letting $\phi = \phi_1$ we have shown $\phi > 0$. Since ϕ solves (5.17) with $\eta = 1$ it follows that $(M, \phi^{2\kappa} g)$ is scalar flat and has a minimal surface boundary. Moreover, since $\phi - 1 \in H^s_\delta$ it follows from Lemma 3.6 and Corollary 3.7 that $(M, \phi^{\frac{4}{n-2}} g)$ is also AE of class H^s_δ. □

To complete the proof of Proposition 5.12 we need to prove a kind of strong maximum principle for rough metrics. We recall that we had constructed a non-negative function ϕ that satisfied an elliptic PDE, and we needed to verify that ϕ vanished nowhere. The main tool we will use to prove this is the weak Harnack inequality of [Tr73] Theorem 5.2, which applies to second order elliptic operators of the form

$$L\,u = \partial_i\left(-a^{ij}\partial_j u + a^i u\right) + b^j\partial_j u + au.$$

The theorem applies under quite general conditions, and certainly holds when a^{ij} is a continuous positive definite symmetric matrix, $a^i \in L^p_{\mathrm{loc}}$, $b^j \in L^p_{\mathrm{loc}}$, and $a \in L^{p/2}_{\mathrm{loc}}$ for some $p > n$. In this case, the weak Harnack inequality states that if $u \in H^1$ satisfies $u \geq 0$ and $Lu \geq 0$ on $B_{5R}(x)$, then

$$||u||_{L^p(B_{2R}(x))} \leq c \inf_{B_R(x)} u$$

where p and c are constants independent of u. In particular, if $u(x_0) = 0$, then u vanishes in a neighbourhood of x_0. If u is continuous and M is connected, as it will be in our applications, we obtain as a consequence that either u is identically zero or it vanishes nowhere.

Now the operator $-\Delta + V$ for a metric $g \in H^s_{\text{loc}}$ and a potential $V \in H^{s-2}_{\text{loc}}$ can be written in the form

$$\partial_i \left(-a^{ij}\partial_j u\right) + b^j \partial_j u + au$$

where $a^{ij} \in H^s_{\text{loc}}$, $b^j \in H^{s-1}_{\text{loc}}$ and $a \in H^{s-2}_{\text{loc}}$. If $s > n/2$ we have from Sobolev embedding

$$a^{ij} \in C^0$$

$$b^j \in L^p_{\text{loc}}$$

where

$$\frac{1}{p} = \frac{1}{2} - \frac{s-1}{n} < \frac{1}{n}$$

since $s > n/2$. If we also have $s \geq 2$, then

$$a \in L^q_{\text{loc}}$$

where

$$\frac{1}{q} = \frac{1}{2} - \frac{s-2}{n} < \frac{2}{n}.$$

So if $s > n/2$ and $s \geq 2$, the weak Harnack inequality can be applied to the operator $-\Delta + V$, at least in the interior of M. In particular $s \geq 2$ when $n > 3$. So it remains to show that we have a weak Harnack inequality when $n = 3$ and in the presence of a boundary. It turns out that both of these cases can be reduced to a situation where the weak Harnack inequality of [Tr73] does apply.

Lemma 5.13 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$, $V \in H^{s-2}_{\rho-2}(M)$, and $\mu \in H^{s-\frac{3}{2}}(\partial M)$. Suppose also that $u \in H^s_{\text{loc}}(M)$ is nonnegative and satisfies*

$$-\Delta u + Vu \geq 0$$

in the interior of M. If $u = 0$ at an interior point of M, then u is identically 0.

Proof: Since u is continuous and M is connected, it is enough to show that the set $\operatorname{int} M \cap u^{-1}(0)$ is open. Suppose x_0 is an interior point and $u(x_0) = 0$. Working in local coordinates about x_0, we know from the discussion above that $-\Delta + V$ can be written in the form

$$-L\,u = \partial_i \left(-a^{ij}\partial_j u + a^i u\right) + b^j \partial_j u + au.$$

where all the coefficients satisfy the conditions of the weak Harnack inequality of [Tr73] except possibly the low order term a which belongs to $H^{s-2}_{\rm loc}$. If $s > 2$ then $a \in L^{p/2}_{\rm loc}$ where

$$\frac{1}{p} = \frac{1}{2} - \frac{s-2}{n} < \frac{2}{n}$$

and hence the low order coefficient also has the correct regularity. So we are only left to consider the case $n = 3$ and $3/2 < s < 2$. Let $\Phi H^s_{\rm loc}$ be a solution of

$$\overline{\Delta}\Phi = V$$

in a neighbourhood of x_0. Here $\overline{\Delta}$ is the Laplacian computed with respect to the flat background metric. It follows that

$$au = \partial_j(\delta^{ij}\partial_i\Phi u) - \delta^{ij}\partial_i\Phi\partial_j u.$$

Now $\partial_i\Phi \in H^{s-1}_{\rm loc}$, and $H^{s-1}_{\rm loc} \subset L^p_{\rm loc}$ where

$$\frac{1}{p} \geq \frac{1}{2} - \frac{s-1}{3} < \frac{1}{3}.$$

So u is a weak supersolution of the operator

$$L\ u = \partial_i\left(-a^{ij}\partial_j u + a^i u\right) + b^j\partial_j u\partial_j(\delta^{ij}\partial_i\Phi u) - \delta^{ij}\partial_i\Phi\partial_j u$$

in a neighbourhood of x_0. All the coefficients of this operator satisfy the hypotheses of the weak Harnack inequality. Since $u \geq 0$ and since $u(x) = 0$, the weak Harnack inequality implies u vanishes in a neighbourhood of x_0. □

Lemma 5.14 *Suppose (M^n, g) is AE of class H^s_ρ with $s > n/2$ and $\rho < 0$, $V \in H^{s-2}_{\rho-2}(M)$, and $\mu \in H^{s-\frac{3}{2}}(\partial M)$. Suppose also that $u \in H^s_{\rm loc}(M)$ is nonnegative and satisfies*

$$-\Delta u + Vu \geq 0$$
$$\partial_\nu u + \mu u \geq 0 \quad \textit{on } \partial M.$$

If $u(x_0) = 0$ at some point $x_0 \in M$, then u vanishes identically.

Proof: From Lemma 5.13 we need only consider the case $x_0 \in \partial M$. Working in local coordinates about x_0 we can do our analysis on $B^+_1(0) \equiv B_1(0) \cap \mathbb{R}^n_+$, where balls are now taken with respect to the flat background metric. Let b be a $H^{s-1}(B_1^+)$ vector field such that $\langle b, \nu\rangle = \mu$ on D_1, where we set $D_1 = \partial B_1^+ \cap B_1$. For example, since $\mu \in H^{s-\frac{3}{2}}(\partial M)$ and $g \in H^s_{\mathrm{loc}}$, we can take $b = \hat\mu\hat\nu$ where $\hat\mu$ is a H^{s-1} extension of μ and $\hat\nu$ is a H^s extension of ν. Integrating by parts, we have for any $\phi \in C^\infty_{\mathrm{c}}(B_1^+ \cup D_1)$

$$\int_{B_1^+} \langle \nabla\phi, b\rangle\, u + \langle b, \nabla u\rangle\, \phi\; dV + \langle \operatorname{div} b, u\phi\rangle_{(B_1^+, g)} = \int_{D_1} \mu\, u\phi\; dA.$$

In particular, if $\phi \geq 0$,

$$\begin{aligned}\int_{B_1^+} \langle \nabla u, \nabla\phi\rangle_g + V\, u\phi +&\\ + \langle \nabla\phi, b\rangle\, u + u\phi\; \operatorname{div} b + \langle b, \nabla u\rangle\, \phi\; dV \quad &= \textstyle\int_{B_1^+} -\phi\Delta u + V u\phi\; dV + \\ &\quad + \textstyle\int_{D_1} \partial_\nu u\phi + \mu\, \phi u\; dA \\ &\quad \geq 0, \end{aligned} \tag{5.18}$$

since u is a supersolution (noting that we have slightly abused notation in (5.18) since the terms involving V, $\operatorname{div} b$, and $\Delta\, u$ are pairings of distributions, not integrals).

To reduce to the interior case, we now construct an elliptic equation on all of B_1. For any function or tensor f defined on $B_1^+(0)$, let $\tilde f$ be the extension of f to B_1 via its push-forward under reflection. These extensions have some limited regularity. In particular, setting

$$\frac{1}{p} = \frac{1}{2} - \frac{s-1}{n} > \frac{1}{n}$$

we have $g \in W^{1,p}(B_1^+)$, $u \in W^{1,p}(B_1^+)$ and $b \in L^p(B_1^+)$. So the reflections satisfy $\tilde g \in W^{1,p}(B_1)$, $\tilde u \in W^{1,p}(B_1)$ and $\tilde b \in L^p(B_1)$. We have to be careful with the low order terms V and $\operatorname{div} b$, however, since these only belong to $H^{s-2}(B_1^+)$. If $s > 2$, then these terms are in $L^{p/2}$ and so are their reflections. On the other hand, if $n = 3$ and $s \in (3/2, 2]$, then $s - 2 \in (-1/2, 0]$. But from Lemma 5.15 we find reflection takes $H^{s-2}(B_1^+)$ to $H^{s-2}(B_1)$ for $s - 2 \in (-1/2, 0]$. Setting $V' = \tilde V + \widetilde{\operatorname{div}} b$ we have $V' \in H^{s-2}(B_1)$ and we obtain from (5.18) and a change of variables argument that

$$\int_{B_1} \langle \nabla\tilde u, \nabla\phi\rangle_{\tilde g} + V'\tilde u\phi + \langle \nabla\phi, \tilde b\rangle_{\tilde g}\, \tilde u + \langle \tilde b, \nabla\tilde u\rangle_{\tilde g}\phi\; \widetilde{dV} \geq 0 \tag{5.19}$$

for all $\phi \in C^\infty_0(B_1)$ with $\phi \geq 0$. Hence $\tilde u$ is a weak supersolution of an operator with coefficients having the regularity required by by [Tr73] except possibly V'. But since $V' \in H^{s-2}(B_1)$ we can argue as before in Lemma 5.13 to convert these terms as well. Since $\tilde{u}(0) = 0$ we obtain that $\tilde{u}$ vanishes in a neighbourhood of 0 and hence u does also. In particular u vanishes in an interior point of M and Lemma 5.13 implies u vanishes identically. $\square$

Lemma 5.15 *Suppose* $u \in H^\sigma(B_1^+)$ *with* $\sigma \in (-\frac{1}{2}, \frac{1}{2})$*, and let* $\tilde{u}$ *be the even reflection of* u *to* B_1*. Then* $u \in H^s(B_1)$*.*

Proof: Since B_1^+ is Lipschitz and since $\sigma \in (-\frac{1}{2}, \frac{1}{2})$, extension by zero is continuous from $H^\sigma(B_1^+)$ to $H^\sigma(\mathbb{R}^n)$. Letting E_0 be the extension operator and R be the reflection diffeomorphism about the surface $x^n = 0$, we have

$$E_0 u + R^* E_0 u \in H^\sigma(\mathbb{R}^n).$$

But $\tilde{u}$ is just the restriction of $E_0 u + R^* E_0 u$ to B_1. $\square$

Chapter 6

APPLICATIONS

6.1 Apparent Horizon Boundary Condition

In Chapter 2 we say CMC conformal method for the apparent horizon boundary problem reduces to solving

$$\begin{aligned} -8\Delta\,\phi - |\sigma|^2\,\phi^{-7} &= 0 \\ 4\partial_\nu\phi + H\phi - \sigma(\nu,\nu)\phi^{-3} &= 0 \quad \text{on } \partial M \end{aligned} \tag{6.1}$$

where σ is a transverse-traceless tensor. We now prove that this equation is solvable so long as

1. (M, g) satisfies $\lambda_g > 0$, $R = 0$ and $H < 0$,

2. σ satisfies $H \leq \sigma(\nu,\nu) \leq 0$.

Theorem 6.1 *Suppose (M^3, g) is AE of class H^s_δ with $s > 3/2$ and $\delta \in (-1, 0)$. Suppose also that $\lambda_g > 0$, $R = 0$, and $H \leq 0$. If $\sigma \in H^{s-1}_{\delta-1}$ is a transverse traceless tensor on M such that $H \leq \sigma(\nu,\nu) \leq 0$ on ∂M, then there exists a conformal factor ϕ solving* (6.1)*. Setting $\hat{g} = \phi^4 g$ and $\hat{K} = \phi^{-2}\sigma$, we have $(M, \tilde{g}, \tilde{K})$ is an AE solution of the constraint equations of class H^s_δ such that ∂M is an apparent horizon and a marginally trapped surface.*

Proof: Let $\phi = 1 + v$, so the Lichnerowicz equation reduces to solving

$$\begin{aligned} -\Delta\,v &= \frac{1}{8}\,|\sigma|^2\,(1+v)^{-7} \\ \partial_\nu v &= -\frac{1}{4}H(1+v) + \frac{1}{4}\sigma(\nu,\nu)(1+v)^{-3} \quad \text{on } \partial M \end{aligned} \tag{6.2}$$

with the constraint $v > -1$. We solve this by means of Proposition 5.9. Since $H \leq \sigma(\nu,\nu)$, we conclude $-\frac{1}{4}H + \frac{1}{4}\sigma(\nu,\nu) \geq 0$. Therefore $v_- = 0$ is a subsolution of (6.2). To find a supersolution, we solve for each $\eta \in [0, 1]$

$$\begin{aligned} -\Delta\,v_\eta &= \frac{1}{8}\,|\sigma|^2 \\ \partial_\nu v_\eta + \frac{\eta}{4}Hv_\eta &= -\frac{\eta}{4}H. \end{aligned} \tag{6.3}$$

The solution exists since $\lambda_g > 0$. We claim moreover that $\phi_\eta = 1 + v_\eta > 0$. Let $I = \{\eta \in [0,1] : \phi_\eta > 0\}$. Arguing as in Proposition 5.12, using the fact that $|\sigma|^2 \geq 0$, we see I is open and nonempty. Suppose $\eta_0 \in \bar{I}$. Then $\phi_{\eta_0} \geq 0$. Since ϕ_η is a supersolution of

$$
\begin{aligned}
-\Delta\, v_\eta &= 0 \\
\partial_\nu v_\eta + \frac{\eta}{4} H v_\eta &= 0,
\end{aligned}
$$

and since ϕ_{η_0} tends to 1 at infinity, Lemma 5.14 then implies $\phi_{\eta_0} > 0$. Hence $\eta_0 \in I$ and I is closed. Let $v_+ = v_1$. We have proved $1 + v_+ > 0$. But then, since

$$
\begin{aligned}
-\Delta\, v_+ &= \frac{1}{8}\,|\sigma|^2 \\
\partial_\nu v_+ &= -\frac{1}{4} H(1 + v_+),
\end{aligned}
\tag{6.4}
$$

and since $-H \geq 0$, $1 + v_1 \geq 0$ and $|\sigma|^2 \geq 0$, we conclude that $v_+ \geq 0$. It follows that

$$
-\Delta\, v_+ = \frac{1}{8}\,|\sigma|^2 \geq \frac{1}{8}\,|\sigma|^2\,(1 + v_+)^{-7}.
$$

Moreover, since $\sigma(\nu,\nu) \leq 0$, we have

$$
\partial_\nu v_+ = -\frac{1}{4} H(1 + v_+) \geq -\frac{1}{4} H(1 + v_+) + \frac{1}{4}\sigma(\nu,\nu)(1 + v_+)^{-3} \quad \text{on } \partial M.
$$

So v_+ is a nonnegative supersolution of (6.2).

Now v_-, v_+, (M, g), and the right hand sides of (6.2) all satisfy the hypotheses of Proposition 5.9. So there exists a nonnegative solution v of (6.2) in H^s_δ. Letting $\tilde{g} = \phi^4 g$ and $\tilde{K} = \phi^{-2}\sigma$, it follows from Lemma 3.6 and Corollary 3.7 that $(M, \tilde{g})$ is AE of class H^s_δ and $\tilde{K} \in H^{s-1}_{\delta-1}$. We also have $(M, \hat{g}, \hat{K})$ solves the constraint equations with apparent horizon boundary condition. To see that the boundary is marginally trapped, we note that

$$
\tilde{H} = \tilde{K}(\tilde{\nu}, \tilde{\nu}) = \phi^{-6}\sigma(\nu,\nu) \leq 0.
$$

Since $\tilde{\theta}_- = \tilde{\theta}_+ + 4\tilde{h}$, we conclude $\tilde{\theta}_- \leq \tilde{\theta}_+ = 0$, and ∂M is marginally trapped. □

6.1.1 Suitable Conformal Data

It is reasonable to ask if any conformal data (M, g, σ) satisfy the hypotheses of Theorem 6.1. We show here that there is, in fact, a large class of suitable conformal data.

Manifolds of the Correct Conformal Class

Suppose we have a smooth, asymptotically Euclidean manifold without boundary (M', g') such that $\lambda_{g'} > 0$. Examples of this include $\mathbb{R}^n$ with the flat metric or any maximal asymptotically Euclidean solution of the constraint equations. From Proposition 5.12 we can assume that $R' = 0$. Let G be the Greens function for the conformal Laplacian on M' with singularity at x. It is well known that when $\lambda_g > 0$, the Greens function for the conformal Laplacian is positive, has a singularity of order r^{-1}, and decays at infinity like r^{-1}.

Let $\phi = 1 + G$ and let $g = \phi^4 g'$ on $M = M' - B_\epsilon(X)$. We wish to show that if ϵ is small enough, then $H > 0$. Now $G = r^{-1} + O(1)$ and $H' = -2r^{-1} + O(1)$. From the conformal change of mean curvature we have

$$\begin{aligned} H &= \phi^{-3} \left(-4\partial_r r^{-1} - 2r^{-1}\frac{1}{r} + O(r^{-1}) \right)\Big|_{r=\epsilon} \\ &= \phi^{-3}\big|_{r=\epsilon} \left(2\epsilon^{-2} + O(\epsilon^{-1}) \right) \\ &> 0 \end{aligned}$$

for ϵ sufficiently small. Repeating this argument (augmenting it in the obvious way to accommodate the boundary) we can remove another small ball from this manifold, and so on to create as many boundary components as we please. Hence there is a rich collection of manifolds with $\lambda_g > 0$.

Gauge Transformation

The condition $R = 0$ and $H \leq 0$ in Theorem 6.1 is not an actual restriction on the choice of metric (whereas $\lambda_g > 0$ certainly is). To see this, we show that every metric with $\lambda_g > 0$ is conformally related to scalar flat metric with negative boundary mean curvature. This fact is related to an observation from [CaB81] for manifolds without boundary that if (M, g) has $\lambda_g > 0$, then it is conformally equivalent to a metric with everywhere positive scalar curvature, everywhere negative scalar curvature, and to a scalar flat metric. This last result is perhaps surprising since the condition $\lambda_g > 0$ on a compact manifold without boundary would preclude a change to a manifold with everywhere negative scalar curvature. The asymptotically Euclidean end allows for the greater flexibility.

Proposition 6.2 *Suppose* (M^3, g') *is AE of class* H^s_δ, $s > 3/2$ *and* $-1 < \delta < 0$. *If* $\lambda_{g'} > 0$, *then there exists a conformal factor* $\phi > 0$ *such that* $\phi - 1 \in H^s_\delta$ *and such that* $(M, g) = (M, \phi^4 g')$ *is scalar flat, has negative boundary mean curvature, and satisfies* $\lambda_g > 0$.

Proof: Since $\lambda_{g'} > 0$, from Proposition 5.12 we can assume without loss of generality that (M, g') satisfies $R' = 0$ and $h' = 0$.

Let $v_\epsilon \in H^s_\delta(M)$ be the unique solution of

$$-\Delta_{g'} v_\epsilon = 0$$
$$\partial_{\nu'} v_\epsilon = -\epsilon.$$

Since $s > 3/2$, v_ϵ depends continuously in C^0_δ on ϵ. Since $v_0 = 0$, we have $v_\epsilon > -1$ for ϵ sufficiently small. Fixing one such $\epsilon > 0$ we have $\phi = 1 + v_\epsilon > 0$. Letting $g = \phi^4 g'$ we see that $R = 0$ and $h = -\epsilon 2 \phi^{-3} < 0$. Since λ_g is a conformal invariant, we have $\lambda_g = \lambda_{g'} > 0$. □

Appropriate Transverse Traceless Tensors

To construct transverse-traceless tensors σ $H \le \sigma(\nu,\nu) \le 0$ we use the Neumann problem for the vector Laplacian. Suppose $\omega \in H^{s-\frac{3}{2}}$ is a one form over the boundary such that $H \le \omega(\nu) \le 0$. From Theorem 5.6 we can solve

$$\Delta_{\mathbb{L}} X = 0$$
$$B_{\mathbb{L}} X = \omega$$

for $X \in H^s_\delta$. Setting $\sigma = \mathbb{L}X$ we have σ is transverse traceless, and $\sigma(\nu,\nu) = \omega(\nu)$ has the desired properties.

Alternatively, suppose $S \in H^{s-1}_{\delta-1}$ is any symmetric, traceless $(0,2)$ tensor with $H \le S(\nu,\nu) \le 0$. Again, from Theorem 5.6, we can solve

$$\Delta_{\mathbb{L}} X = \operatorname{div} S$$
$$B_{\mathbb{L}} X = 0$$

Setting $\sigma = S - \mathbb{L}X$ we have $\operatorname{div}\sigma = 0$ and $\sigma(\nu,\nu) = S(\nu,\nu)$. So σ is a transverse-traceless tensor with $H \le \sigma(\nu,\nu) \le 0$ as required.

6.1.2 Open Problems

There remains several interesting questions concerning the apparent horizon boundary problem. Our construction provides a sufficient condition on the conformal data (M, g, σ) to yield a solution of the constraints. But we have not found necessary and sufficient conditions, as is possible on manifolds without boundaries. There are two questions that need to be addressed. First: is the condition $lambda_g > 0$ necessary? It is for manifolds without boundary, but we have not proved that it is for the apparent horizon boundary problem. The second, and more significant, problem to to find a replacement for the condition $H \leq \sigma(\nu, \nu)$. This is not a conformally invariant inequality, and a more subtle interaction between the metric and $\sigma(\nu, \nu)$ needs to be found.

Another question arises in connection with the difference between the results of [Ma03] and [Da03]. From these papers, we know we can find solutions of the constraint equations such that either

$$\theta_- \leq \theta_+ = 0 \qquad \text{(implying } H \leq 0\text{)}$$
$$\theta_+ \leq \theta_- \leq 0 \qquad \text{(implying } H \geq 0\text{)}$$

Neither construction allows one to find surfaces with $\theta_- \leq \theta_+ < 0$. We can find one-parameter families of initial data starting with the construction in [Ma03] and terminating with the construction in [Da03], but only by passing through the condition $\theta_+ = \theta_- = H = 0$. We would like to understand better the relationship between the two constructions by determining first if one can construct solutions with $\theta_- \leq \theta_+ < 0$ (i.e. trapped surfaces with $h \leq 0$), and second if we can pass between the two constructions without going through the condition $H = 0$.

Finally, there is interest in the numerical relativity community in finding a construction on compact manifolds with boundary. Here the asymptotically Euclidean ends are truncated and a boundary condition is placed on the new ends. So the boundary of M is divided into two components, one that satisfies the apparent horizon boundary condition, and one that satisfies a replacement for the asymptotically Euclidean condition. There already exist in the literature substitutions for the asymptotically Euclidean condition (e.g. see [YP82] for the conformal factor and [O'92] for the transverse traceless tensor). It remains to be seen if these can be coupled with the apparent horizon condition on the inner boundary.

6.2 Rough Initial Data

We have already seen in Section 6.1 that it is possible to construct initial data in $H^s_{\rm loc}$ with $s > 3/2$. We isolate the result for manifolds without boundary here since we can show $\lambda_g > 0$ is both necessary and sufficient for the Lichnerowicz equation

$$-8\Delta\,\phi - |\sigma|^2\,\phi^{-7} = 0 \tag{6.5}$$

to be solvable. We also provide the approximation theorem required by [KR].

Theorem 6.3 *Suppose (M^3, g) is AE of class H^s_δ with $s > 3/2$ and $\delta \in (-1, 0)$. Let σ be any transverse-traceless tensor in $H^{s-1}_{\delta-1}(M)$. There exists a conformal factor ϕ solving* (6.5) *if and only if $\lambda_g > 0$. Moreover, if a solution exists then it is unique.*

Proof: If a solution exists, then it follows form the Hamiltonian constraint that g is conformally related to a metric with non-negative scalar curvature, and from Proposition 5.12 that $\lambda_g > 0$.

If $\lambda_g > 0$ we can assume without loss of generality that $R = 0$. Setting $\phi = 1 + v$, solving the Lichnerowicz equation is equivalent to solving

$$-8\Delta\,v = |\sigma|^2\,(1+v)^{-7} \tag{6.6}$$

with the constraint $v > -1$.

Evidently $v_- = 0$ is a subsolution of (6.6). To find a supersolution, we solve

$$-8\Delta\,v_+ = |\sigma|^2\,.$$

From the weak maximum principle we find $v_+ \geq v_- = 0$. We can now apply Proposition 5.9 to find there exists a nonnegative solution v of (6.6) in H^s_δ.

Turning to uniqueness, suppose v_1 and v_2 are two solutions. Then

$$-8\Delta\,(v_1 - v_2) = |\sigma|^2\,((1+v_1)^{-7} - (1+v_2)^{-7}). \tag{6.7}$$

Fixing $\epsilon > 0$ we have $(v_1 - v_2 - \epsilon)^{(+)} = 0$ if $v_1 \leq v_2 + \epsilon$. On the other hand, if $v_1 > v_2 + \epsilon$, then $(1+v_1)^{-7} - (1+v_2)^{-7} < 0$. So multiplying (6.7) by $(v_1 - v_2 - \epsilon)^{(+)}$ and integrating we have

$$-8\int_M (v_1 - v_2 - \epsilon)^{(+)}\Delta\,(v_1 - v_2) \leq 0.$$

Integrating by parts we obtain, exactly as in the proof of Lemma 5.8, that $v_1 \leq v_2 + \epsilon$. Since $\epsilon > 0$ is arbitrary, $v_1 \leq v_2$ and from symmetry we obtain $v_1 = v_2$. $\square$

6.2.1 Approximation by Smooth Solutions

The following theorem shows that every solution of the constraints constructed in Theorem 6.3 can be approximated arbitrarily well by smooth solutions.

Theorem 6.4 *Let (M^3, g_0, K_0) be a maximal AE solution of the constraint equations of class H^s_δ with $s > 3/2$ and $\delta \in (-1, 0)$. For any $\epsilon > 0$, there exists a maximal AE solution $(M, g_\epsilon, K_\epsilon)$ of the constraint equations of class H^t_δ for every $t \geq s$ such that $||g_0 - g_\epsilon||_{H^s_\delta} < \epsilon$ and $||K_0 - K_\epsilon||_{H^{s-1}_{\delta-1}} < \epsilon$.*

Proof: Let $\{g_k\}_{k=1}^{\infty}$ be a sequence of metrics on M in H^t_δ for every $t \geq s$ such that $||g_k - g_0||_{H^s_\delta} \to 0$. We will write $\Delta^k_{\mathbb{L}}$, $\mathbb{L}^k$, and div_k for the differential operators corresponding to g_k.

To construct a sequence of transverse-traceless tensors, we let $\{S_k\}_{k=1}^{\infty}$ be an arbitrary sequence of traceless $(0,2)$-tensors in $H^{t-1}_{\delta-1}$ for every $t \geq s$ converging to K_0 in $H^{s-1}_{\delta-1}$. Let $X_k \in H^s_\delta$ be the unique solution of $\Delta^k_{\mathbb{L}} X_k = \operatorname{div}_k S_k$. Since $\operatorname{div}_k S_k \in H^{t-2}_{\delta-2}$ for every $t \geq s$, it follows that $\mathbb{L}X_k \in H^{t-1}_{\delta-1}$ for every $t \geq s$. Since $\Delta_{\mathbb{L}}$ is invertible, we have uniform bounds on the norm of the inverse of $\Delta^k_{\mathbb{L}}$. Hence

$$||X_k||_{H^s_\delta} \lesssim ||\operatorname{div}_k S_k|| \lesssim ||\operatorname{div}_k||_{H^{s-1}_{\delta-1}} ||S_k - K_0||_{H^{s-1}_{\delta-1}} + ||\operatorname{div}_k - \operatorname{div}||_{H^{s-1}_{\delta-1}} ||K_0||_{H^{s-1}_{\delta-1}}.$$

So $||X_k||_{H^s_\delta} \to 0$. Letting $\sigma_k = S_k - \mathbb{L}^k X_k$ it follows that σ_k is transverse-traceless with respect to g_k, and is in $H^{t-1}_{\delta-1}$ for every $t \geq s$. Moreover, $||\sigma_k - K_0||_{H^{s-1}_{\delta-1}} \leq ||\mathbb{L}^k X_k||_{H^{s-1}_{\delta-1}} + ||S_k - K||_{H^{s-1}_{\delta-1}} \to 0$.

The correction to g_k is now accomplished by the implicit function theorem. Let

$$\mathcal{F}(g, \sigma, v) = -8\Delta_g v + R_g(1+v) - |\sigma|^2_g (1+v)^{-7}.$$

For fixed g and σ, let $\mathcal{F}_{g,\sigma}(v) = \mathcal{F}(g, \sigma, v)$. Using Lemma 6.5 proved below, we find that the Fréchet derivative of $\mathcal{F}_{g,\sigma}$ is

$$\mathrm{d}\,\mathcal{F}_{g,\sigma}(v)(h) = -a\Delta_g h + R_g h + 7\,|\sigma|^2_g (1+v)^{-8} h,$$

and $\mathrm{d}\,\mathcal{F}_{g,\sigma}(v)$ is hence continuous in a neighbourhood of (g, σ, v) for each v with $v > -1$. Moreover,

$$\mathrm{d}\,\mathcal{F}_{g_0,K_0}(0)(h) = L(h) = -a\Delta_{g_0} h + R_{g_0} h + 7\,|K_0|^2\, h.$$

Since R_0 is nonnegative, L is an isomorphism. Since $\mathcal{F}(g_0, K_0, 0) = 0$, and since $g_k \to g_0$ and $\sigma_k \to K_0$, the implicit function theorem (e.g. [AP93] Lemma 2.2.1) implies that for k sufficiently large there exists $v_k \in H^s_\delta$ such that $v_k \to 0$ and such that $\mathcal{F}(g_k, \sigma_k, v_k) = 0$. From the equation $\mathcal{F}(g_k, \sigma_k, v_k) = 0$ and a bootstrap we have $v_k \in H^t_\delta$ for every $t \geq s$. Letting $\tilde{g}_k = (1 + v_k)^4 g_k$ and $\tilde{K}_k = (1 + v_k)^{-2}\sigma_k$, we conclude from Lemma 3.6 and Corollary 3.7 that $(M, \tilde{g}_k, \tilde{K}_k)$ is an AE data set of class H^σ_δ for every $\sigma \geq s$ and $(\tilde{g}_k - g_0, \tilde{K}_k - K_0)$ converges to 0 in $H^s_\delta \times H^{s-1}_{\delta-1}$. Taking k sufficiently large proves the theorem. $\square$

The following lemma completes the proof of Theorem 6.4.

Lemma 6.5 *Suppose* (M^3, g) *is AE of class* H^s_ρ *with* $s > 3/2$ *and* $\rho < 0$*, and suppose* $\sigma \in H^{s-1}_{\delta-1}$ *with* $\delta \in (-1, 0)$*. Let* U *be the open subset* $\{v \in H^s_\delta : v > -1\}$*, and let* $\mathcal{G} : U \to H^{s-2}_{\delta-2}$ *be given by*

$$\mathcal{G}(v) = |\sigma|^2 (1 + v)^{-7}.$$

Then $\mathcal{G}$ *has a Fréchet derivative* $\mathrm{d}\,\mathcal{G}$ *given by*

$$\mathrm{d}\,\mathcal{G}(v)(h) = -7\,|\sigma|^2 (1 + v)^{-8} h.$$

Proof: We first consider the maps

$$g(v) = (1 + v)^{-7}$$
$$g'(v) = -7(1 + v)^{-8}.$$

Since $1 \in H^s_\epsilon$ for every $\epsilon > 0$, it follows from Lemma 3.6 that g and g' are continuous as maps from U to H^s_ϵ. Since $1 + v > 0$, it follows that there exists a ball B_r of radius r in H^s_δ such that $1 + v + h > 0$ for all $h \in B_r$. Taking $h \in B_r$, it follows from the continuity of g' that the map $t \to g'(v + th)h$ from $[0, 1]$ to H^s_ϵ is Bochner integrable. Since evaluation at a point is a continuous linear functional on H^s_ϵ, we have from the properties of the Bochner integral

$$\left(\int_0^1 g'(v + th)h\,dt\right)(x) = \int_0^1 g'(v(x) + th(x))h(x)\,dt = g(v(x) + h(x)) - g(v(x)).$$

Hence

$$g(v + h) - g(v) = \int_0^1 g'(v + th)h\,dt.$$

Now

$$
\begin{aligned}
||g(v+h)-g(v)-g'(v)h||_{H^s_\epsilon} &= \left|\left|\int_0^1 \big(g'(v+th)-g'(v)\big)\,h\,dt\right|\right|_{H^s_\epsilon} \\
&\leq \int_0^1 \big|\big|\big(g'(v+th)-g'(v)\big)\,h\big|\big|_{H^s_\epsilon}\,dt \\
&\leq \int_0^1 \big|\big|\big(g'(v+th)-g'(v)\big)\big|\big|_{H^s_\epsilon}\,dt\,||h||_{H^s_\delta}.
\end{aligned}
$$

Since $|\sigma|^2 \in H^{s-2}_{2\delta-2}$, we can take $\epsilon < -\delta$ to obtain

$$
||\mathcal{G}(v+th)-\mathcal{G}(v)-\mathrm{d}\,\mathcal{G}(v)h||_{H^{s-2}_{\delta-2}} \lesssim ||\sigma^2||_{H^{s-2}_{2\delta-2}} \int_0^1 \big|\big|\big(g'(v+th)-g'(v)\big)\big|\big|_{H^s_\epsilon}\,dt\,||h||_{H^s_\delta}.
$$

Since g' is continuous in a neighbourhood of v, it follows that

$$
\int_0^1 \big|\big|\big(g'(v+th)-g'(v)\big)\big|\big|_{H^s_\epsilon}\,dt
$$

can be made arbitrarily small by taking $||h||_{H^s_\delta}$ small. We conclude that $\mathrm{d}\,\mathcal{G}(v)$ is the Fréchet derivative of $\mathcal{G}$ at v. $\square$

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