Inverse Methods for Reconstructing Basal Boundary Data

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III-Posessed Boundary Value Problem

We wish to determine conditions at the base $B$ of a glacier from measurements at the surface $S$. We consider the simplified case of flow through a cross section $E$ under the hypothesis that the in-plane velocity components and all out-of-plane gradients are zero. Assuming Glen’s flow law (Glen, 1955) for the material stress-strain relationship, the equation for momentum balance in the out of plane direction can be written in terms of the out of plane velocity component $u$:

$$\nabla \cdot (\rho \mathbf{v}^2 u^2) = (2\lambda/3)\dot{\gamma} u$$

(1)

where $\rho$ is the density of ice, $\dot{\gamma}$ the acceleration due to gravity, and $u$ the out-of-plane surface slope, and where the flow rate parameter $A$ and the exponent $n$ are empirical constants (we use $n = 3$ in our study). We assume that shear stresses vanish at the surface:

$$\nabla \cdot (\rho \mathbf{v}^2 u^2) = 0$$

(2)

We also assume that velocities are known on $S$:

$$u|_S = u_S$$

(3)

Problem [N-D]: Find a solution of (1) satisfying (2) and (3).

III-Posessed Auxiliary Problems

Boundary conditions (2) and (3) uniquely determine a solution of (1), but problem [N-D] is ill-posed. Given a smooth surface field $u_S$, there need not be a solution (although there will be a solution for an infinitesimally perturbed surface velocity field). More significantly, tiny errors in $u_S$ lead to uncontrollably large errors in the corresponding solution of (1). We must use regularization methods to determine approximate solutions, with the quality of the approximation depending nonlinearly on an estimate for our uncertainty in $u_S$ (and in our model (3)).

Well-Posessed Auxiliary Problems

Well-posed boundary value problems for (1) require a boundary condition at the base $B$. Our regularization techniques work with three such conditions:

Ditrichlet (4)

$$u|_B = u_B$$

Neumann (5)

$$\nabla \cdot (\rho \mathbf{v}^2 u^2) - \sigma_B = 0$$

Robin (6)

$$\nabla \cdot (\rho \mathbf{v}^2 u^2) - \gamma u + u_B = 0$$

$$\nabla \cdot (\rho \mathbf{v}^2 u^2) - \gamma u + u_B = 0$$

(7)

These conditions are used in the following well-posed boundary-value problems:

Problem [D-N]: Find a solution of (1) satisfying (4) on $B$ and (2) on $S$.

Problem [N-D]: Find a solution of (1) satisfying (5) on $B$ and (3) on $S$.

Problem [R-N]: Find a solution of (1) satisfying (6) on $B$ and (3) on $S$.

Problem [D-N]: Find a solution of (1) satisfying (4) on $B$ and (2) on $S$.

Problem [N-D]: Find a solution of (1) satisfying (5) on $B$ and (3) on $S$.

Problem [R-N]: Find a solution of (1) satisfying (6) on $B$ and (3) on $S$.

We have implemented three regularization methods using as adapting known algorithms for the linear ($n = 1$) version of Problem [N-D] in the inner loop of an iterative procedure to find approximate solutions of the nonlinear problem.

Kozlov-Maz’ya Iteration

Kozlov and Maz’ya (1990) introduced a regularization technique for linear Problem [N-D] that consists of alternately solving Problems [D-N] and [R-D]. The algorithm starts with an initial guess $u_B^0$ for the basal velocities and continues until the solution of [D-N] matches the surface data $u_S$ to within a specified tolerance that reflects uncertainty in the data and the model.

Dependence on Domain Geometry

Reconstruction of basal velocities in domains with parabolic cross sections using Kozlov-Maz’ya iteration. Cross sections have depth equal to 1 and widths $W$ with (a) $W = 1$ (b) $W = 2$, and (c) $W = 3$. Synthetic data generated using a Dirichlet condition at the base specifying a sliding region at the center (dashed line). Noise (2% of peak velocity for a frozen base) was added prior to reconstruction.

Comparison of Methods

Reconstruction of basal velocities in a paraboloid domain using (a) Kozlov-Maz’ya, (b) constrained Ditrichlet, and (c) constrained Robin algorithm. All algorithms find solutions that are consistent with the noisy data. Constrained Ditrichlet detects the near frozen regions best. Kozlov-Maz’ya is easily the most efficient of the algorithms (solves linear forward problems solved with a quadratic programming), but suffers from oscillations near the edges.

Applicability in Unfavorable Domains

Even in domains where resolution at the base is limited, much of the interior solution can be reconstructed: (a) Contours of synthetic velocities with sliding on the right side of the base. (b) Reconstruction using Kozlov-Maz’ya iteration. (c) Reconstruction of velocities at the base: Kozlov-Maz’ya (orange), constrained Ditrichlet (green), constrained Robin (purple), true (black dashed).

References


We carried out a real data inversion using the Kozlov-Maz’ya iteration applied to surface measurements from Raymond (1971). (a) Contours derived from measurements (Raymond, 1971). (b) Synthetical data (b) and (c) used to reconstruct basal shear stress. Kozlov-Maz’ya (orange), constrained Ditrichlet (green), constrained Robin (purple). Reconstruction depends highly on the value of $\alpha$ in (In 1). Athabasca data are not consistent with transverse flow with Glen parameters $n = 3$ and $A = 0$.