Exercise Oprea 2.2.8: Suppose M is a surface which is the level set of a function g, $g(x, y, z) = c$. Show that $\nabla g(p)$ is a normal vector for M at every $p \in M$.

Solution:
Let $\mathbf{v} \in T_pM$. Let α be a curve in M with $\alpha(0) = p$ and $\alpha'(0) = v$. Note that since α is a curve in M, and since g is constant on M, $g(\alpha(t))$ is constant for all t. Hence

$$\mathbf{v}[g] = \left.\frac{d}{dt}\right|_{t=0} g(\alpha(t)) = 0.$$

On the other hand, on the previous homework we showed that

$$\frac{d}{dt} g(\alpha(t)) = \nabla g(\alpha(t)) \cdot \alpha'(t).$$

Evaluating this at $t = 0$, using the facts that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$ we have

$$0 = \mathbf{v}[g] = \nabla g(p) \cdot v.$$

Since $v \in T_p(M)$ is arbitrary, $\nabla g(p)$ is normal to M at p.