1. Suppose V is a finite-dimensional inner-product space with inner product $\langle\cdot,\cdot\rangle$ and that $T : V \to V$ is symmetric. Show that T has no complex eigenvalues.

Hint: Let W be the complex vector space of vectors of the form $a + ib$ where $a, b \in V$. You need not show that this is a vector space. We extend T to a map $T : W \to W$ by

$$T(a + ib) = Ta + iTb.$$

It's easy to see that this map is complex linear; don't show this. If the characteristic equation $\det(T - \lambda I)$ has a complex root λ, then there is a vector $z = x + iy$ in W such that $T(x + iy) = \lambda(x + iy)$; there is a little work to be done here, but don't do it either unless you are looking for some extra fun.

Extend the inner product to complex vectors in W by

$$\langle a + ib, w\rangle = \langle a, w\rangle + i\,\langle b, w\rangle \qquad \text{and} \qquad \langle w, a + ib\rangle = \langle w, a\rangle + i\,\langle w, b\rangle\,.$$

Now show

1. $T\overline{z} = \overline{\lambda}\overline{z}$ if z is an eigenvector with eigenvalue λ.
2. $< Tz, w >= \langle z, Tw\rangle$ for all complex vectors z and w.

From these two ingredients you can show that any eigenvalue of T must be real by looking at $\langle Tz, \overline{z}\rangle$.

Solution:
We suppose $z = x + iy$ is a complex eigenvector of T with eigenvalue λ. Note that

$$T(\overline{z}) = T(x - iy) = Tx - iTy = \overline{Tx + iTy} = \overline{T(x + iy)} = \overline{\lambda z} = \overline{\lambda}\overline{z}.$$

Now suppose $v, w \in W$. Then $v = a + ib$ and $w = c + id$ for some $a, b, c, d \in V$. Then

$$\begin{aligned}\langle Tv, w\rangle = \langle T(a + ib), c + id\rangle &= \langle Ta + iTb, c + id\rangle \\ &= \langle Ta, c\rangle + i\,\langle Ta, d\rangle + i\,\langle Tb, c\rangle - \langle Tb, d\rangle\,.\end{aligned}$$

Using the symmetry of T acting on V we find

$$\begin{aligned}\langle Tv, w\rangle &= \langle a, Tc\rangle + i\,\langle a, Td\rangle + i\,\langle b, Tc\rangle - \langle b, Td\rangle \\ &= \langle a + ib, Tc + iTd\rangle \\ &= \langle v, Tw\rangle\,.\end{aligned}$$

Now consider

$$\langle Tz, \overline{z}\rangle = \langle \lambda z, \overline{z}\rangle = \lambda < z, \overline{z} >\,.$$

But

$$\langle Tz, \overline{z}\rangle = \langle z, T\overline{z}\rangle = \left\langle z, \overline{\lambda}\overline{z}\right\rangle = \overline{\lambda}\,\langle z, \overline{z}\rangle\,.$$

Hence

$$\lambda \langle z, \overline{z} \rangle = \overline{\lambda} \langle z, \overline{z} \rangle .$$

One readily verifies that $\langle z, \overline{z} \rangle = \langle x, x \rangle + \langle y, y \rangle \neq= 0$. So

$$\lambda = \overline{\lambda}$$

and λ is real.

2. Let A and B be $n \times n$ matrices. Show $\operatorname{tr} AB = \operatorname{tr} BA$. Conclude that if $T : V \to V$ is a linear map between finite dimensional vector spaces that $\operatorname{tr} T$ is well-defined.

Solution:
Let $A = [A_i j]$ and $B = [B_i j]$ so that

$$AB_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Then

$$\operatorname{tr} AB = \sum_{i=1}^{n} AB_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} B_{ki} A_{ik} = \operatorname{tr} BA.$$

Now if A is any $n \times n$ matrix and P is any invertible $n \times n$ matrix,

$$\operatorname{tr} P^{-1} A P = \operatorname{tr} A P P^{-1} = \operatorname{tr} A.$$

Now suppose $T : V \to V$ is linear and V is finite dimensional. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases on V, and let P be the change of basis matrix introduced in class. Then

$$T_{\mathcal{B}_2} = P^{-1} T_{\mathcal{B}_1} P.$$

But then

$$\operatorname{tr} T_{\mathcal{B}_2} = \operatorname{tr} P^{-1} T_{\mathcal{B}_1} P = \operatorname{tr} T_{\mathcal{B}_1} P P^{1} = \operatorname{tr} T_{\mathcal{B}_1}.$$

Hence the trace of $T_{\mathcal{B}}$ is independent of the choice of basis $\mathcal{B}$ and can be used to define $\operatorname{tr} T$.

3. Let $\mathcal{B}$ be a basis for $T_p M$.

a) Show that $I_{\mathcal{B}} S_{\mathcal{B}} = II_{\mathcal{B}}$, where $S = S_p$ is the shape operator on $T_p M$.

b) Explain why $I_{\mathcal{B}}$ is an *invertible* 2×2 matrix.

c) Show that $S_{\mathcal{B}} = I_{\mathcal{B}}^{-1} II_{\mathcal{B}}$.

d) Conclude that

$$K = \frac{ln - m^2}{EG - F^2}$$

and

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)}.$$

Solution, part a:
Let $\mathcal{B} = \{x_1, x_2\}$ be a basis for T_pM. Let $II_\mathcal{B} = [II_{ij}]$, $I_\mathcal{B} = [I_{ij}]$, and $S_\mathcal{B} = [S_{ij}]$. Then

$$II_{ij} = II(x_i, x_j) = \left\langle Sx_i, x_j \right\rangle = \left\langle x_i, Sx_j \right\rangle = \left\langle x_i, x_k S_{kj} \right\rangle = \langle x_i, x_k \rangle S_{kj} = II_{ik} S_{kj}.$$

Hence

$$II_\mathcal{B} = I_\mathcal{B} S_\mathcal{B}.$$

Solution, part b:
Suppose to the contrary that $I_\mathcal{B}$ is not invertible. Then there is a nonzero vector v such that

$$II_{ij} v_j = 0$$

for each i. So

$$v_i II_{ij} v_j = 0$$

Let $w = v_1 x_1 + v_2 x_2$ so that $w_\mathcal{B} = v$. Then $\langle w, w \rangle = v^t II_\mathcal{B} v = 0$. Hence $w = 0$ and therefore $v = 0$. This is a contradiction.

Solution, part c:
This follows immediately from parts (a) and (b).

Solution, part d:
Using a chart to determine the usual basis x_u and x_v for T_pM we write

$$I_\mathcal{B} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \qquad II_\mathcal{B} = \begin{bmatrix} l & m \\ m & n \end{bmatrix}.$$

We see that

$$K = \det S = (\det I_\mathcal{B})^{-1} \det II_\mathcal{B} = (ln - m^2)/(EG - F^2).$$

Also

$$I_\mathcal{B}^{-1} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix}.$$

So

$$I_\mathcal{B}^{-1} II_\mathcal{B} = \frac{1}{EG - F^2} \begin{bmatrix} Gl - Fm & Gm - Fn \\ -Fm + El & -Fm + En \end{bmatrix}.$$

Consequently

$$H = \frac{1}{2} \operatorname{tr} I_\mathcal{B}^{-1} II_\mathcal{B} = \frac{GL + En - 2Fm}{2(EG - F^2)}$$

4. Write a Maple procedure **UnitNormal** that takes an expression for a chart $\mathbf{x}$ and the name of two coordinate variables (e.g. u and v) and returns a simplified expression for the unit normal in terms of u and v. Verify your procedure works by computing the unit normal of the helicoid $\mathbf{x}(u, v) = (v \cos u, v \sin u, bv)$. (You have already computed this normal on a previous homework.)

5. Write a Maple procedure **FundamentalForms** that takes an expression for a chart **x** and the name of two coordinate variables (e.g. u and v) and returns two 2×2 matrices, the matrices of the first and second fundamental forms computed with respect to the basis $\{x_u, x_v\}$.

6. Compute E, F, and G, as well as l, m, and n for the chart $\mathbf{x}(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$. You may use Maple to assist you in your computation. Explain why your results for F and G make intuitive sense.

Solution:
See worksheet for the computation that gives $F = 0$ and $G = 1$. These make sense because the lines of latitude and longitude are orthogonal, and because we parameterized lines of longitude with a unit speed curve.

7. Let $\mathbf{x}$ be a chart with domain D into the surface M. Let $\tilde{\alpha} : [0, T] \to D$ be a curve in D, so $\tilde{\alpha}(t) = (u(t), v(t))$ for some functions u and v. Let $\alpha(t) = \mathbf{x}(\alpha(t))$, so α is a curve in M. Show that

$$L(\alpha) = \int_0^T E(\tilde{\alpha}(t))(u'(t))^2 + 2 * F(\tilde{\alpha}(t))u'(t)v'(t) + G(\tilde{\alpha}(t))(v'(t))^2 \, dt.$$

This shows us that E, F, and G can be used to compute the lengths of curves in local coordinates.

Let $\mathbf{x}$ be the chart in problem 6. Let $\tilde{\alpha}(t) = (t, \pi/4)$ where $-\pi < t < \pi$. Use the formula developed above and the values of E, F, and G computed in problem 6 to compute the length of α. Explain why the result of this computation makes intuitive sense.

Solution:
Note that $\alpha'(t) = \mathbf{x}_u(\alpha(t))u'(t) + \mathbf{x}_v(\alpha(t))v'(t)$. Hence

$$|\alpha'(t)|^2 = \langle \mathbf{x}_u(\alpha(t))u'(t) + \mathbf{x}_v(\alpha(t))v'(t), \mathbf{x}_u(\alpha(t))u'(t) + \mathbf{x}_v(\alpha(t))v'(t) \rangle = E(u')^2 + 2Fu'v' + G(v')^2.$$

Hence

$$L(\alpha) = \int_0^T |\alpha'(t)| \, dt = \int_0^T \sqrt{E(\tilde{\alpha}(t))(u'(t))^2 + 2 * F(\tilde{\alpha}(t))u'(t)v'(t) + G(\tilde{\alpha}(t))(v'(t))^2} \, dt.$$

For the sphere example we have $E = \sin^2 v$, $F = 0$, $G = 1$, $u'(t) = 1$, and $v'(t) = 0$. So

$$L = \int_{-\pi}^{\pi} \sqrt{\sin^2 \pi/4} \, dt = 2\pi \sin \pi/4 = 2\pi \frac{\sqrt{2}}{2}.$$

This result makes sense because the curve on the sphere is the line of latitude for angle $\pi/4$ which is a circle of radius $\cos \pi/4$.

8. Write a Maple procedure that computes the Gaussian and mean curvatures of a surface. Then compute the Gaussian and mean curvatures of the helicoid.

Solution:
See worksheet.

9. Write a Maple procedure **Shape** that takes an expression for a chart $\mathbf{x}$ and the name of two coordinate variables (e.g. u and v) and returns the matrix of the shape operator with respect to the basis $\mathbf{x}_u$ and $\mathbf{x}_v$.

Solution:
See worksheet.

10. Use the program you wrote in the previous problem to re-do Exercise 2.2.16 in Oprea.

Solution:
See worksheet.