1. Suppose V is a finite-dimensional inner-product space with inner product $\langle \cdot, \cdot \rangle$ and that $T : V \to V$ is symmetric. Show that T has no complex eigenvalues.

Hint: Let W be the complex vector space of vectors of the form $a + ib$ where $a, b \in V$. You need not show that this is a vector space. We extend T to a map $T : W \to W$ by

$$T(a + ib) = Ta + iTb.$$

It's easy to see that this map is complex linear; don't show this. If the characteristic equation $\det(T - \lambda I)$ has a complex root λ, then there is a vector $z = x + iy$ in W such that $T(x + iy) = \lambda(x + iy)$; there is a little work to be done here, but don't do it either unless you are looking for some extra fun.

Extend the inner product to complex vectors in W by

$$\langle a + ib, w \rangle = \langle a, w \rangle + i \langle b, w \rangle \qquad \text{and} \qquad \langle w, a + ib \rangle = \langle w, a \rangle + i \langle w, b \rangle .$$

Now show

1. $T\bar{z} = \bar{\lambda}\bar{z}$ if z is an eigenvector with eigenvalue λ.
2. $< Tz, w >= \langle z, Tw \rangle$ for all complex vectors z and w.

From these two ingredients you can show that any eigenvalue of T must be real by looking at $\langle Tz, \bar{z} \rangle$.

2. Let A and B be $n \times n$ matrices. Show $\operatorname{tr} AB = \operatorname{tr} BA$. Conclude that if $T : V \to V$ is a linear map between finite dimensional vector spaces that $\operatorname{tr} T$ is well-defined.

3. Let $\mathcal{B}$ be a basis for T_pM.

a) Show that $I_{\mathcal{B}} S_{\mathcal{B}} = II_{\mathcal{B}}$, where $S = S_p$ is the shape operator on T_pM.

b) Explain why $I_{\mathcal{B}}$ is an *invertible* 2×2 matrix.

c) Show that $S_{\mathcal{B}} = I_{\mathcal{B}}^{-1} II_{\mathcal{B}}$.

d) Conclude that

$$K = \frac{ln - m^2}{EG - F^2}$$

and

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)}.$$

4. Write a Maple procedure **UnitNormal** that takes an expression for a chart $\mathbf{x}$ and the name of two coordinate variables (e.g. u and v) and returns a simplified expression for the unit normal in terms of u and v. Verify your procedure works by computing the unit normal of the helicoid $\mathbf{x}(u, v) = (v \cos u, v \sin u, bu)$. (You have already computed this normal on a previous homework.)

5. Write a Maple procedure **FundamentalForms** that takes an expression for a chart **x** and the name of two coordinate variables (e.g. u and v) and returns two 2×2 matrices, the matrices of the first and second fundamental forms computed with respect to the basis $\{x_u, x_v\}$.

6. Compute E, F, and G, as well as l, m, and n for the chart $\mathbf{x}(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$. You may use Maple to assist you in your computation. Explain why your results for F and G make intuitive sense.

7. Let **x** be a chart with domain D into the surface M. Let $\tilde{\alpha} : [0,T] \to D$ be a curve in D, so $\tilde{\alpha}(t) = (u(t), v(t))$ for some functions u and v. Let $\alpha(t) = \mathbf{x}(\alpha(t))$, so α is a curve in M. Show that

$$L(\alpha) = \int_0^T E(\tilde{\alpha}(t))(u'(t))^2 + 2 * F(\tilde{\alpha}(t))u'(t)v'(t) + G(\tilde{\alpha}(t))(v'(t))^2 \, dt.$$

This shows us that E, F, and G can be used to compute the lengths of curves in local coordinates.

Let **x** be the chart in problem 6. Let $\tilde{\alpha}(t) = (t, \pi/4)$ where $-\pi < t < \pi$. Use the formula developed above and the values of E, F, and G computed in problem 6 to compute the length of α. Explain why the result of this computation makes intuitive sense.

8. Write a Maple procedure that computes the Gaussian and mean curvatures of a surface. Then compute the Gaussian and mean curvatures of the helicoid.

9. Write a Maple procedure **Shape** that takes an expression for a chart **x** and the name of two coordinate variables (e.g. u and v) and returns the matrix of the shape operator with respect to the basis $\mathbf{x}_u$ and $\mathbf{x}_v$.

10. Use the program you wrote in the previous problem to re-do Exercise 2.2.16 in Oprea.