All parts of this homework to be completed in Maple should be done in a single worksheet. You can submit either the worksheet by email or a printout of it with your homework.

1. Let $k(s)$ be a smooth function on $\mathbb{R}$. Let

$$\theta(s) = \int_0^s k(u)\, du$$

and

$$\alpha(s) = \left(\int_0^s \cos(\theta(u))\, du, \int_0^s \sin(\theta(u))\, du \right).$$

Show that α is a smooth unit speed curve with signed curvature $\kappa_p(s) = k(s)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a translation (so $T(x) = x + v$ for some constant vector v). Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection $S(x, y) = (-x, y)$. If α is a smooth plane curve, show that the signed curvatures of $T \circ \alpha$ and $R \circ \alpha$ are the same as those of α, but the signed curvature of $S \circ \alpha$ is the negative of the signed curvature of α. Also, show that each of $T \circ \alpha$, $R \circ \alpha$, and $S \circ \alpha$ are all unit speed curves if α is.

3. Suppose α and β are unit speed plane curves defined on the same interval $I = [a, b]$ such that $\alpha(a) = \beta(a)$, $\alpha'(a) = \beta'(a)$ and such that their two curvatures agree at every point in I. Show that $\alpha = \beta$.

Hint: Let T_α and N_α be the tangent and planar normal to α, and use similar notation for β. Let

$$D = T_\alpha \cdot T_\beta + N_\alpha \cdot N_\beta.$$

Explain why $D = 2$ if and only if $T_\alpha = T_\beta$ and $N_\alpha = N_\beta$. Now show that $D' \equiv 0$ and $D(a) = 2$. Now use the fact that $T_\alpha \equiv T_\beta$ to conclude that $\alpha = \beta$.

Conclude that planar curvature determines a plane curve up to an (oriented) rigid motion (i.e. a composition of a translation and a rotation).

4. Use Maple to plot the trace of a plane curve with signed curvature $\kappa_p(s) = 2\cos(s)$. Explain why $\alpha(2\pi)$ lies on the x-axis.

5. Let α be a unit speed plane curve. Its center of curvature is

$$\epsilon(s) = \alpha(s) + \frac{1}{\kappa_p(s)} N_p(s).$$

a) Show that the circle centered at $\epsilon(s)$ is tangent to α at $\alpha(s)$ and has the same curvature as α at that point. You should use facts you know about the curvature of a circle.

b) The curve $\epsilon(s)$ is called the evolute of α. Assume that $\kappa' < 0$ for the curve. Show that the unit tangent to ϵ is $N_p(s)$ and the signed unit normal to ϵ is $-T$.

c) Let v be the arclength parameter of ϵ. Show that

$$\frac{dv}{ds} = \left| \frac{\kappa_p'(s)}{\kappa^2} \right|$$

d) Compute the signed curvature of $\epsilon(s)$.

6. Exercise 1.3.23.

7. (This problem to be done entirely with Maple.) Viviani's curve is defined by

$$\alpha(t) = (\cos(t)^2 - 1/2, \sin(t)\cos(t), \sin(t)).$$

a) Show that α lies on the sphere of radius 1 centered at $(-1/2, 0, 0)$ and on the cylinder $x^2 + y^2 = 1/2$.

b) Make a plot in Maple to demonstrate that α lies on this sphere. The commands **plots[spacecurve]**, **plottools[sphere]** and **plottools[display]** might come in handy. Also note that if you end a line in Maple with a colon rather than a semicolon, the output will be suppressed, which is handy for things like the output of **plottools[sphere]**.

c) Compute the curvature and torsion of Viviani's curve.

d) Verify that the curvature and torsion of Viviani's curve satisfy the formula

$$R^2 = (1/\kappa)^2 + ((1/\kappa)'(1/\tau))^2$$

from the previous problem.