1. Let γ be a unit speed curve into a surface M with tangent vector T. Suppose the surface is orientable, and has unit normal U. At any point on the curve where $T \equiv \dot{\gamma} \neq 0$ we can decompose

$$\ddot{\gamma} = aT + b(U \times T) + cU$$

for unique constants a,b,c. Moreover, $a = 0$ since $||\dot{\gamma}||^2$ is constant. Recall that the curvature κ of γ in $\mathbb{R}^3$ is

$$\kappa = ||\ddot{\gamma}|| = \sqrt{b^2 + c^2}.$$

Thus we can write

$$\ddot{\gamma} = \kappa \cos(\phi)(U \times T) + \kappa \sin(\phi)U$$

for some angle ϕ, uniquely defined up to multiples of 2π. We called the quantity

$$\kappa_g = \kappa \cos(\phi)$$

the geodesic curvature of γ and

$$\kappa_n = \kappa \sin(\phi)$$

the normal curvature of γ. The point of this exercise is to generalize the notion of geodesic curvature a little and connect it to the notion of covariant derivatives. You should note that geodesic curvature, as defined, is a signed quantity, like planar curvature. We can do this because the surface is orientable, and hence we can make a global choice of a unit tangent vector perpendicular to T.

a) Suppose γ is a not-necessarily unit speed curve in M. We define the geodesic and normal curvatures of γ to be the corresponding curvatures of a unit speed reparameterization of γ. Show that normal and geodesic curvatures of γ can be computed via

$$k_n = \frac{\langle \gamma'', U \rangle}{\langle \gamma', \gamma' \rangle}$$

$$k_g = \frac{\langle \gamma'', U \times \gamma' \rangle}{[\langle \gamma', \gamma' \rangle]^{3/2}}.$$

Hint: Recall that a unit speed reparameterization β of γ satisfies $\beta(s(t)) = \gamma(t)$ where s is the arclength function of γ.

b) Let γ be a regular but not-necessarily unit speed curve in M. Show that

$$\gamma'' = \mu' T + k_g \mu^2 (U \times T) + k_n \mu^2 U$$

where T is the unit tangent to γ and $\mu = |\gamma'|$.

c) Recall that we defined that a curve γ is a geodesic if its acceleration is always in the normal direction. Show that this implies γ is a geodesic if and only if γ' is parallel along γ.

d) Demonstrate that

$$\nabla^M_{\gamma} \gamma' = \mu' T + k_g \mu^2 (U \times T)$$

and that γ is a geodesic if and only if γ is constant speed and κ_g is zero along γ.

Solution, part a:
Let $\beta(s)$ be a unit speed reparameterization of γ so $\beta(s(t)) = \gamma(t)$ where $s(t)$ is an arclength function. That is, $s'(t) = |\gamma'(t)|$. Since β is unit speed,

$$\beta''(s) = \kappa_g(U \times T) + \kappa_n U.$$

On the other hand,

$$\begin{aligned}\gamma''(t) &= \frac{d}{dt^2}\beta(s(t)) \\ &= \beta''(s)s'^2 + \beta'(s)s'' \\ &= \beta''(s)s'^2 + Ts''.\end{aligned}$$

Hence

$$\begin{aligned}\langle \gamma''(t), U\rangle &= \langle \beta''(s), U\rangle (s')^2 + \langle T, U\rangle s'' \\ &= \left\langle \kappa_g(U \times T) + \kappa_n U, U\right\rangle s'^2 + 0 \\ &= \kappa_g \langle (U \times T), U\rangle (s')^2 + \kappa_n \langle U, U\rangle s'^2 \\ &= \kappa_n s'^2\end{aligned}$$

since U, T, and $U \times T$ are all orthonormal. Hence

$$\kappa_n = \frac{\langle \gamma''(t), U\rangle}{s'^2} = \frac{\langle \gamma''(t), U\rangle}{|\gamma'|^3}.$$

Moreover, we similarly have

$$\langle \gamma''(t), U \times T\rangle = \langle \beta''(s), U \times T\rangle (s')^2 + \langle T, U \times T\rangle s'' = \kappa_g s'^2.$$

Hence

$$\kappa_g = \frac{\langle \gamma''(t), U \times T\rangle}{s'^2} = \frac{\langle \gamma''(t), U \times (\gamma'/s')\rangle}{s'^2} = \frac{\langle \gamma''(t), U \times \gamma'\rangle}{s'^3} = \frac{\langle \gamma''(t), U \times \gamma'\rangle}{|\gamma'|^{3/2}}.$$

Solution, part b:
We can write

$$\gamma'' = aT + b(U \times T) + cU.$$

From part (a) we know that

$$c = \langle \gamma'', U\rangle = \kappa_n |\gamma'|^2 = \mu^2 \kappa_n,$$

and

$$c = \langle \gamma'', U \times T\rangle = \langle \gamma'', U \times \gamma'\rangle (1/\mu) = k_g \mu^3 / mu = k_g \mu^2.$$

Finally,

$$\langle \gamma', \gamma'\rangle = \mu^2$$

so

$$2\langle \gamma'', \gamma' \rangle = 2\mu\mu'$$

and

$$\langle \gamma'', T \rangle = \langle \gamma'', \gamma' \rangle (1/\mu) = \mu'.$$

Hence

$$\gamma'' = \mu' T + k_g \mu^2 (U \times T) + k_n \mu^2 U.$$

Solution, part c:
Recall that $\nabla_\gamma^M \gamma'$ is the projection of $(d/dt)\gamma'(t)$ into the tangent space of M. Since

$$\gamma'' = \mu' T + k_g \mu^2 (U \times T) + k_n \mu^2 U.$$

this occurs precisely when

$$\mu' T + k_g \mu^2 (U \times T) = 0$$

and hence

$$\gamma'' = k_n \mu^2 U.$$

This last condition if and only if γ is a geodesic.

Solution, part d:
From the previous problem, γ is a geodesic if and only if

$$\mu' T + k_g \mu^2 (U \times T) = 0.$$

Since T and $U \times T$ are linearly independent, and since $\mu \neq 0$, this occurs exactly when μ' and $k_g = 0$ along the curve, i.e. the curve is constant speed with vanishing geodesic curvature.

2. Show that if v and w are vector fields defined along γ that $(v \cdot w)' = v \cdot \nabla_\gamma^M w + w \cdot \nabla_\gamma^M v$.

Solution:
Note that

$$v' = \nabla_\gamma^M v + cU$$

for some function c defined along γ. Similarly,

$$w' = \nabla_\gamma^M w + dU.$$

Hence

$$\begin{aligned}(v \cdot w)' &= v \cdot w' + v' \cdot w \\ &= v \cdot (\nabla_\gamma^M w + dU) + (\nabla_\gamma^M v + cU) \cdot w.\end{aligned}$$

Since $v \cdot U$ and $w \cdot U = 0$ we conclude

$$(v \cdot w)' = v \cdot \nabla_\gamma^M w + w \cdot \nabla_\gamma^M v$$

3. Suppose γ is a regular curve in the orientable surface M with normal U. Suppose v is parallel transported along γ. Show that $v^\perp = U \times v$ is also parallel transported along γ, and that $\{v, v^\perp\}$ is an orthonormal basis along γ.

Solution:
To show that $U \times v$ is parallel transported it is enough to show that

$$\frac{d}{dt} U \times v = cU$$

for some function c defined along γ. We note that

$$\frac{d}{dt} U \times v = U' \times v + U \times v'.$$

Now v is a parallel transported, so $v' = dU$ for some function d along γ. But then $U \times v' = d(U \times U) = 0$. Moreover, since $U\dot{U} = 1$,

$$U'\dot{U} = 0$$

and U' is everywhere a tangent vector. Hence $U' \times v$ is a cross product of two tangent vectors and is hence everywhere normal. That is,

$$U' \times v = cN$$

for some function c. So $(U \times v)' = cN$ and $U \times v$ is parallel.

Suppose at some point p along the curve that v has unit length. Then $U \times v$ is orthonormal to v and $(v, U \times v)$ is then an orthonormal basis for the tangent space T_pM. We recall that parallel transport preserves lengths and angles. Hence v and $U \times v$ remain orthonormal along the curve, and for an orthonormal basis at each point along the curve.

4. With the same notation as in the previous problem, at each point on the curve we can write $\dot{\gamma} = \mu(\cos(\theta)v + \sin(\theta)v^\perp)$, where μ is the speed of the curve and θ is a function uniquely defined up to multiples of 2π. Show that $\theta' = \mu\kappa_g$. In particular, for unit speed curves, $\kappa_g = \dot{\theta}$.

Solution:
Since v and $v^\perp$ for an orthonormal basis along the curve, we can write

$$\dot{\gamma} = |\dot{\gamma}|(av + bv^\perp)$$

where $a^2 + b^2 = 1$. So we can find an angle function θ such that $a = \cos(\theta)$ and $b = \sin(\theta)$ along the curve. Hence

$$\dot{\gamma} = \mu(\cos(\theta)v + \sin(\theta)v^\perp).$$

But then

$$\begin{aligned}\nabla^M_\gamma \dot{\gamma} &= \dot{\mu}(\cos(\theta)v + \sin(\theta)v^\perp) + \mu\dot{\theta}(-\sin(\theta)v + \cos\theta)v^\perp + \mu(\cos(\theta)\nabla^M_\gamma v + \sin(\theta)\nabla^M_\gamma v^\perp) \\ &= \dot{\mu}(\cos(\theta)v + \sin(\theta)v^\perp) + \mu\dot{\theta}(-\sin(\theta)v + \cos\theta)v^\perp\end{aligned}$$

since v and $v^\perp$ are parallel transported along the curve. Note that $T = \cos(\theta)v + \sin(\theta)v^\perp$ and

$$U \times T = U \times (\cos(\theta)v + \sin(\theta)v^\perp) = \cos(\theta)U \times v + \sin(\theta)U \times (U \times v) = \cos(\theta)v^\perp - \sin(\theta)v.$$

So

$$\nabla^M_\gamma \dot\gamma = \mu' T + \mu\theta' U \times T.$$

From problem 1,

$$\nabla^M_\gamma \dot\gamma = \mu' T + k_g \mu^2 (U \times T).$$

Hence $k_g \mu = \theta'$.

5.

a) Show that for all $n \in \mathbb{Z}$, $\lim_{x\to 0^+} x^n e^{-1/x^2} = 0$.

b) Define $\phi(x) = e^{-1/x^2}$ for $x > 0$ and $\phi(x) = 0$ for $x \leq 0$. Show that ϕ is differentiable to all orders at 0 and that all its derivatives vanish there. Hence ϕ is a smooth function.

c) Construct a smooth function that is even, non-negative, equals 1 at 0, and vanishes outside of $(-1,1)$.

d) Construct a smooth function that is non-negative, has values in $[0,1]$, is equal to zero on $(-\infty,0)$ and is equal to 1 on $(1,\infty)$.

Solution, part a:
We show

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{/} x^n = 0$$

for all $n \in \mathbb{N}$. By change of variable it is enough to show

$$\lim w \to \infty \frac{w^{n/2}}{e^w} = 0,$$

and indeed it is enough to establish this only when n is even, i.e. that

$$\lim w \to \infty \frac{w^k}{e^w} = 0,$$

for all $k \in \mathbb{N}$. But this last fact follows from l'Hopital's rule, (with k derivatives applied to both the numerator and denominator) to obtain

$$\lim w \to \infty \frac{w^k}{e^w} = \lim w \to \infty \frac{k!}{e^w} = 0.$$

Solution, part b:
We claim that for each $n \in \mathbb{N}$ that there exist polynomials p and q such that

$$\phi^{(n)}(x) = \begin{cases} e^{-x^2} \frac{p(x)}{q(x)} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

The claim is obvious when $n = 1$. Suppose the claim holds for $n \in \mathbb{N}$. Then $\phi^{(n+1)}$ also has the desired form for $x \neq 0$ as well and it suffices to show that ϕ^n is differentiable at $x = 0$ and that the derivative is 0. Clearly

$$\lim_{x\to 0^-} \frac{\phi^n(x)}{x} = 0.$$

On the other hand,

$$\lim_{x\to 0^+} \frac{\phi^n(x)}{x} = \lim_{x\to 0^+} \frac{e^{-1/x^2} p(x)}{x q(x)}.$$

Let M be the order of the polynomial $xq(x)$. Then

$$\lim_{x\to 0^+} \frac{e^{-1/x^2} p(x)}{x q(x)} = \lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^M} p(x) \frac{x^M}{x q(x)}$$

By part (b),

$$\lim_{x\to 0^+} \frac{e^{-1/x^2}}{x^M} = 0$$

and since the remaining terms are bounded, the limit is zero.

6. Suppose γ is a curve from p to q and $\hat{\gamma}$ is a reparameterization of γ. Show that $\Pi^{\hat{\gamma}}_{pq} = \Pi^{\gamma}_{pq}$

Solution:
We suppose γ has domain $[a,b]$ and that $\sigma : [c,d] \to [a,b]$ is a function with $\sigma(c) = a$, $\sigma(d) = b$, and $\hat{\gamma} = \gamma \circ \sigma$. Let $V \in T_{\gamma(a)}M$, and let let v be the a parallel vector field along γ, so $v(t) \in T_{\gamma(t)}M$ for each $t \in [a,b]$. So $\Pi^{\gamma}_{p,q}(V) = v(b)$.

We claim that $\hat{v}(t) = v(\sigma(t))$ is parallel along $\hat{\gamma}$. If this is established, then since $\hat{v}(c) = v(\sigma(c)) = v(a) = V$ and $\hat{v}(d) = v(b)$ it follows that

$$\Pi^{\hat{\gamma}}_{pq} V = \hat{v}(d) = v(b) = \Pi^{\gamma}_{pq} V.$$

Since V is arbitrary, the linear maps are the same.

Certainly $\hat{v}(t) = v(\sigma(t)) \in T_{\gamma(\sigma(t))}M = T_{\hat{\gamma}(t)}M$ for each $t \in [c,d]$, so $\hat{v}$ is a vector field along $\hat{\gamma}$. Moreover

$$\frac{d}{dt}\hat{v} = v'(\sigma(t))\sigma'(t).$$

Since v is parallel, $v'(s) = c(s)U(\gamma(s))$ for some function c defined along γ. Hence

$$v'(\sigma(t)) = c(\sigma(t))U(\gamma(\sigma(t))) = \hat{c}(t)U(\hat{\gamma}(t))$$

where $\hat{c} = c \circ \sigma$. So

$$\frac{d}{dt}\hat{v} = \hat{c}(t)\sigma'(t)U(\hat{\gamma}(t))$$

and therefore $\hat{v}$ is parallel along $\hat{\gamma}$.

7. Compute the inverse of Π^{γ}_{pq} by explicitly constructing a curve β with $\Pi^{\beta}_{pq} = (\Pi^{\gamma}_{pq})^{-1}$

Solution:
We suppose $\gamma : [a, b] \to M$. We then define $\beta : [a, b] \to M$ by

$$\beta(t) = \gamma(a + b - t).$$

That is, $\beta = \gamma \circ \sigma$ where $\sigma(t) = a + b + t$. Suppose $V \in T_p(M)$ and let $v(t)$ be its parallel transport along γ. Let $w(t) = v \circ \sigma$. Then $w(a) = v(\sigma(a)) = v(b) = \Pi^{\gamma}_{pq} V$. Arguing as in the previous problem we see that w is parallel along β and hence

$$\Pi^{\beta}_{qp} \Pi^{\gamma}_{pq} V = w(b) = v(\sigma(b)) = v(a) = V.$$

So Π^{β}_{qp} is a left inverse of Π^{γ}_{pq}. But then, since the maps are linear between finite dimensional vector spaces, it is a right inverse as well.