1. From the equation $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = 0$, show

$$\begin{aligned} ln - m^2 = \partial_v \left[\langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \mathbf{x}_v \rangle \right] - \partial_u \left[\langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \mathbf{x}_v \rangle \right] \\ - \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v \rangle + \langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v \rangle . \end{aligned} \tag{1}$$

Since all quantities on the right-hand side of (1) can be computed from knowledge of E, F, and G alone, we concluded that one can compute Gauss curvature from the first fundamental form.

Solution:
Observe that

$$\langle (\mathbf{x}_{uu})_v, \mathbf{x}_v \rangle = \partial_v \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle - \langle \mathbf{x}_{uu}, \mathbf{x}_{vv} \rangle .$$

Since $\langle U, \mathbf{x}_v \rangle = 0$,

$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \mathbf{x}_v \rangle .$$

Moreover, since $\langle U, \mathbf{x}_u \rangle = 0$, $\langle U, \mathbf{x}_v \rangle = 0$, and $\langle U, U \rangle = 1$ we have

$$\langle \mathbf{x}_{uu}, \mathbf{x}_{vv} \rangle = \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v \rangle + A_{uu}A_{vv}.$$

Since $A_{uu} = l$ and $A_{vv} = n$ we conclude

$$\langle (\mathbf{x}_{uu})_v, \mathbf{x}_v \rangle = \partial_v \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \mathbf{x}_v \rangle - \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v \rangle - ln.$$

A similar computation shows

$$\langle (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = \partial_u \left[\langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \mathbf{x}_v \rangle \right] - \langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v \rangle - m^2.$$

Subtracting these equations and using the fact that $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = 0$ we arrive at the desired equation.

2. Without computing anything new, write down an analogous formula to (1) that would be obtained from the equation $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, \mathbf{x}_u \rangle = 0$.

Solution:
We obtain this equation by swapping the roles of u and v. This has the effect of also interchanging E and G as well as l and n. We obtain

$$\begin{aligned} nl - m^2 = \partial_u \left[\langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^u_{vv}\mathbf{x}_u, \mathbf{x}_u \rangle \right] - \partial_v \left[\langle \Gamma^v_{vu}\mathbf{x}_v + \Gamma^u_{vu}\mathbf{x}_u, \mathbf{x}_u \rangle \right] \\ - \langle \Gamma^v_{vv}\mathbf{x}_v + \Gamma^u_{vv}\mathbf{x}_u, \Gamma^v_{uu}\mathbf{x}_v + \Gamma^u_{uu}\mathbf{x}_u \rangle + \langle \Gamma^v_{vu}\mathbf{x}_v + \Gamma^u_{vu}\mathbf{x}_u, \Gamma^v_{vu}\mathbf{x}_v + \Gamma^u_{vu}\mathbf{x}_u \rangle . \end{aligned} \tag{2}$$

3. Compute a formula analogous to (1) that would be obtained from

$$\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_u \rangle = 0.$$

This one requires actual computation.

Solution:
We have

$$\langle (\mathbf{x}_{uu})_v, \mathbf{x}_u \rangle = \partial_v \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle - \langle \mathbf{x}_{uu}, \mathbf{x}_{uv} \rangle = \frac{1}{2}\partial_v \partial_u \langle \mathbf{x}_u, \mathbf{x}_u \rangle - \langle \mathbf{x}_{uu}, \mathbf{x}_{uv} \rangle .$$

We also have

$$\langle (\mathbf{x}_{uv})_u, \mathbf{x}_u\rangle = \partial_u \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle = \frac{1}{2}\partial_u\partial_v \langle \mathbf{x}_u, \mathbf{x}_u\rangle - \langle \mathbf{x}_{uv}, \mathbf{x}_{uv}\rangle .$$

Hence

$$\langle \mathbf{x}_{uu})_v - \mathbf{x}_{uv})_u, \mathbf{x}_u\rangle = \frac{1}{2}\partial_v\partial_u \langle \mathbf{x}_u, \mathbf{x}_u\rangle - \frac{1}{2}\partial_u\partial_v \langle \mathbf{x}_u, \mathbf{x}_u\rangle = \frac{1}{2}(E_{vu} - E_{uv}) = 0.$$

4. Show that $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, U\rangle = 0$ implies the equation

$$l_v - m_u = \Gamma^u_{uv} l + (\Gamma^u_{uv} - \Gamma^u_{uu})m - \Gamma^v_{uu} n$$

This is one of the two Codazzi-Mainardi equations. The other Codazzi-Mainardi equation comes from $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, U\rangle = 0$. For extra credit, write down what this other equation is. (Do no hard work).

Solution:
We have

$$\begin{aligned}
\langle (\mathbf{x}_{uu})_v, U\rangle &= \partial_v \langle \mathbf{x}_{uu}, U\rangle - \langle \mathbf{x}_{uu}, U_v\rangle \\
&= \partial_v l - \langle \mathbf{x}_{uu}, U_v\rangle \\
&= l_v - \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, U_v\rangle \\
&= l_v - \Gamma^u_{uu} \langle \mathbf{x}_u, -S\mathbf{x}_v\rangle - \Gamma^v_{uu} \langle \mathbf{x}_v, -S\mathbf{x}_v\rangle \\
&= l_v + \Gamma^u_{uu} m - \Gamma^v_{uu} n.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle (\mathbf{x}_{uv})_u, U\rangle &= \partial_u \langle \mathbf{x}_{uv}, U\rangle - \langle \mathbf{x}_{uv}, U_u\rangle \\
&= m_u + \Gamma^u_{uv} l + \Gamma^v_{uv} n.
\end{aligned}$$

Hence

$$l_v - m_u = \Gamma^u_{uv} l + (\Gamma^v_{uv} - \Gamma^v_{uu})m - \Gamma^v_{uu} n.$$

The other equation comes from swapping the roles of u and v to obtain

$$n_u - m_v = \Gamma^v_{vu} n + (\Gamma^u_{uv} - \Gamma^u_{vv})m - \Gamma^u_{vv} l.$$

5. Let f, g, and h be functions of u and v. Show that

$$\frac{1}{\sqrt{gh}}\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{gh}}\right) = \frac{f_{vv}}{gh} - \frac{1}{2}\frac{f_v g_v}{g^2 h} - \frac{1}{2}\frac{f_v h_v}{gh^2}.$$

Solution:
We have

$$\begin{aligned}
\frac{1}{\sqrt{gh}}\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{gh}}\right) &= \frac{1}{\sqrt{gh}}\left[\frac{f_{vv}}{\sqrt{gh}} - \frac{1}{2} f_v \frac{g_v h + g h_v}{(gh)^{3/2}}\right] \\
&= \frac{f_{vv}}{gh} - \frac{1}{2}\frac{f_v g_v}{g^2 h} - \frac{1}{2}\frac{f_v h_v}{gh^2}.
\end{aligned}$$

6. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Compute all the Christoffel symbols for this chart in terms of E and G and their derivatives.

Solution:
The equations for the Chrystoffel symbols become trivial to solve if $F = 0$ and we have

$$\Gamma^u_{uu} = \frac{1}{2}\left(\frac{E_u}{E}\right) \qquad \Gamma^u_{uv} = \frac{1}{2}\left(\frac{E_v}{E}\right) \qquad \Gamma^u_{vv} = -\frac{1}{2}\left(\frac{G_u}{E}\right)$$
$$\Gamma^v_{uu} = -\frac{1}{2}\left(\frac{E_v}{G}\right) \qquad \Gamma^v_{uv} = \frac{1}{2}\left(\frac{G_u}{G}\right) \qquad \Gamma^v_{vv} = \frac{1}{2}\left(\frac{G_v}{G}\right)$$

7. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Use (1) and the previous two problems to show that

$$K = -\frac{1}{2}\frac{1}{\sqrt{EG}}\left[\partial_v\left(\frac{E_v}{\sqrt{EG}}\right) + \partial_u\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

Solution:
Note that if $F = 0$, then $K = (ln - m^2)/(EG)$. Hence (1) can be written (using the fact that $\langle \mathbf{x}_v, \mathbf{x}_u \rangle = F = 0$,

$$K \cdot EG = \partial_v\left[\Gamma^v_{uu}G\right] - \partial_u\left[\Gamma^v_{uv}G\right] - \Gamma^u_{uu}\Gamma^u_{vv}E - \Gamma^v_{uu}\Gamma^u_{vv}G + \Gamma^u_{uv}\Gamma^u_{uv}E + \Gamma^v_{uv}\Gamma^v_{uv}G.$$

Using our computations of the Chyrstoffel symbols in the previous problem we have

$$K \cdot EG = -\frac{1}{2}\partial_v E_v - \frac{1}{2}\partial_u G_u + \frac{1}{4}\frac{E_u G_u}{E} + \frac{1}{4}\frac{E_v G_v}{G} + \frac{1}{4}\frac{E_v E_v}{E} + \frac{G_u G_u}{G}.$$

Hence

$$K = -\frac{1}{2}\left[\frac{E_{vv}}{EG} - \frac{1}{2}\frac{E_v G_v}{EG^2} - \frac{1}{2}\frac{E_v E_v}{E^2 G}\right] - \frac{1}{2}\left[\frac{G_{uu}}{EG} - \frac{1}{2}\frac{G_u G_u}{EG^2} - \frac{1}{2}\frac{G_u E_u}{E^2 G}\right].$$

By Problem 4 applied twice we conclude that

$$K = -\frac{1}{2}\frac{1}{\sqrt{EG}}\left[\partial_v\left(\frac{E_v}{\sqrt{EG}}\right) + \partial_u\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

8. Suppose for some chart $\mathbf{x}$ that $E = 1$ and $F = 0$ everywhere. Show that

$$K = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial u^2}\sqrt{G}.$$

Solution:
We note that

$$\partial_u\sqrt{G} = \frac{1}{2}\frac{1}{\sqrt{G}}G_u = \frac{1}{2}\frac{G_u}{\sqrt{G}}.$$

Now if $E = 1$ and $F = 0$ everywhere, then our computation from problem 6 reads

$$K = -\frac{1}{2}\frac{1}{\sqrt{G}}\partial_u\left[\frac{G_u}{\sqrt{G}}\right] = -\frac{1}{G}\partial_u\left[\frac{1}{2}\frac{G_u}{\sqrt{G}}\right] = -\frac{1}{\sqrt{G}}\partial_u^2\sqrt{G}.$$

9. Consider the surface of revolution $\mathbf{x}(u,v) = (f(u)\cos(v), g(u), f(u)\sin(v))$ where $(f')^2 + (g')^2 = 1$. Write down what E, F, and G are for this chart and compute K from E, F, and G.

Solution:
We have already computed for such a chart that $E = 1$, $F = 0$, and $G = (f(u))^2$. By the previous problem then,

$$K = -\frac{1}{f(u)}\partial_u^2 f(u) = -\frac{f''(u)}{f(u)}.$$