Some review:

In class we defined the Christoffel symbols via the equation

$$\mathbf{x}_{\alpha\beta} = \Gamma^u_{\alpha\beta}\mathbf{x}_u + \Gamma^v_{\alpha\beta}\mathbf{x}_v + A_{\alpha\beta}U$$

where α and β are either u or v and

$$[A_{\alpha\beta}] = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

We showed that the Christoffel symbols can be computed by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^u_{\alpha\beta} \\ \Gamma^v_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{\alpha\beta}, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_{\alpha\beta}, \mathbf{x}_v \rangle \end{pmatrix}.$$

We also computed the right-hand sides of the previous equations:

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2}E_u \qquad \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2}E_v \qquad \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - \frac{1}{2}G_u$$
$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2}E_v \qquad \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2}G_u \qquad \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2}G_v.$$

The following exercises give an alternate proof that Gaussian curvature can be computed from the first fundamental form alone. It is more direct than the proof I gave in class, but it gives you little idea of what it is the bugs measure when they measure Gaussian curvature. Nevertheless, we can use this method to obtain nice explicit formulas for Gaussian curvature without introducing an orthonormal frame.

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1. From the equation $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = 0$, show

$$\begin{aligned} ln - m^2 = {} & \partial_v \left[\langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \mathbf{x}_v \rangle \right] - \partial_u \left[\langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \mathbf{x}_v \rangle \right] \\ & - \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v \rangle + \langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v \rangle . \end{aligned} \tag{1}$$

Since all quantities on the right-hand side of (1) can be computed from knowledge of E, F, and G alone, we concluded that one can compute Gauss curvature from the first fundamental form.

2. Without computing anything new, write down an analogous formula to (1) that would be obtained from the equation $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, \mathbf{x}_u \rangle$.

3. Compute a formula analogous to (1) that would be obtained from

$$\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_u \rangle = 0.$$

This one requires actual computation.

4. Show that $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, U \rangle = 0$ implies the equation

$$l_v - m_u = \Gamma^u_{uv} l + (\Gamma^u_{uv} - \Gamma^u_{uu})m - \Gamma^v_{uu} n$$

This is one of the two Codazzi-Mainardi equations. The other Codazzi-Mainardi equation comes from $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, U \rangle = 0$. For extra credit, write down what this other equation is. (Do no hard work).

5. Let f, g, and h be functions of u and v. Show that

$$\frac{1}{\sqrt{gh}}\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{gh}}\right) = \frac{f_{vv}}{gh} - \frac{1}{2}\frac{f_v g_v}{g^2 h} - \frac{1}{2}\frac{f_v h_v}{gh^2}.$$

6. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Compute all the Christoffel symbols for this chart in terms of E and G and their derivatives.

7. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Use (1) and the previous two problems to show that

$$K = -\frac{1}{2}\frac{1}{\sqrt{EG}}\left[\partial_v\left(\frac{E_v}{\sqrt{EG}}\right) + \partial_u\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

8. Suppose for some chart $\mathbf{x}$ that $E = 1$ and $F = 0$ everywhere. Show that

$$K = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial u^2}\sqrt{G}$$

9. Consider the surface of revolution $\mathbf{x}(u, v) = (f(u)\cos(v), g(u), f(u)\sin(v))$ where $(f')^2 + (g')^2 = 1$. Write down what E, F, and G are for this chart and compute K from E, F, and G.