Please see the rules on the second page.

1. Oprea 1.4.16

2. We showed in class that if a surface has trivial holonomy, then it is flat. The converse is false. Find, with proof, a surface that is flat but that has nontrivial holonomy. That is, find a point p on your flat surface and a curve from p to p such that parallel transport along γ does not result in the identity map. You should identify (perhaps without rigorous proof) all of the rotations that are possible by holonomy for your surface.

3. A simple closed plane curve encloses a region with area A. Show there is a point on the curve where the unsigned curvature k satisfies $k \le \sqrt{\pi/A}$. Find a curve that encloses an area A and satisfies $k \ge \sqrt{\pi/A}$ everywhere. Feel free to use the Umlaufsatz.

4. Consider the family of surfaces $S_a = \{(x, y, z) : z = x^2 + ay^2\}$. Show that for $a \neq a'$, the surfaces S_a and $S_{a'}$ are diffeomorphic but that there does not exist an isometry from S_a to $S_{a'}$. Your answer should clearly indicate that you understand the definition of a diffeomorphism.

5. Use the Clairaut relation to give a qualitative description of all the geodesics on the torus. Be sure to discuss all cases of values of the angular momentum.

6. Suppose for some chart $\mathbf{x}$ that $E = G = e^{\lambda(u,v)}$ and $F = 0$, where λ is a smooth function of u and v. Do a computation by hand to show the surface's Gauss curvature satisfies

$$\Delta\lambda + 2Ke^{\lambda} = 0$$

where Δ is the Laplacian $\partial_u^2 + \partial_v^2$.

7. Let M be a compact surface. Suppose S is not diffeomorphic to a sphere. Prove that there are points in the surface with positive, negative, and zero Gauss curvature.

8. Let α be a unit speed curve with principal normal N and binormal B. The *tube* about α of radius a is the surface

$$\mathbf{x}(u, v) = \alpha(u) + a\left[N(u)\cos(v) + B(u)\sin(v)\right].$$

   a) Show that $\mathbf{x}$ is regular if a is less than the reciprocal of curvature, κ^{-1}, of α at all points.

   b) Show that the area of the region $u_0 < u < u_1$ and $-\pi < v < \pi$ is

$$2\pi a(u_1 - u_0).$$

   c) Show that the parameter curves of constant u are geodesics.

9. A *tangent developable* of a unit speed curve α is the surface

$$\mathbf{x}(u, v) = \alpha(u) + v\alpha'(u).$$

   a) Show that $\mathbf{x}$ is regular so long as $\kappa_\alpha > 0$ and $v \neq 0$.

b) Show that every tangent developable is (locally) isometric to the plane. Hint: Build a chart into the plane that has the same first fundamental form.

c) Use Maple to graph the tangent developable of the helix $\alpha(t) = (\cos(t), \sin(t), t)$.

Rules and format:

- You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
- If you find a suspected typo, please contact me as soon as possible and I will communicate it to the class if needed.
- You may not discuss the exam with anyone else until after the due date/time.
- You are permitted to reference Oprea or any other differential geometry text you like. If you use another text, you must cite it when you use it.
- Each problem is weighted equally (10 points each).
- The due date/time is absolutely firm.
- We will schedule a hints session for this exam.