Induction is used to prove that a collection of statements $P(k)$ depending on $k \in \mathbb{N}$ are all true. A statement is simply a mathematical phrase that must be either true or false. Here are some examples of statements $P(k)$ that depend on a natural number; some of these use definitions we haven't formally introduced yet, but you should understand them anyways.

Examples of $P(k)$:

- $k \geq 1$.

- $k^2 > k + 10$.

- k is even.

- k is either even or odd.

- $\sum_{j=1}^{k} j = n(n+1)/2$.

- k is a product of prime numbers.

- $5k < -1$.

- $\sum_{j=1}^{k} j = 10$.

In all of these cases $P(k)$ can be either true or false. For example, in the last case, $P(1)$ is false because the sum is 1 and $1 \neq 10$, but $P(4)$ is true because $1 + 2 + 3 + 4 = 10$.

Exercise 1:   For each statement in the list above, determine (without proof) the natural numbers for which the statement is true.

The version of induction we have used is the following.

Theorem 2.17 (Weak Induction):   Suppose $P(k)$ is a statement depending on $k \in \mathbb{N}$. To show that $P(k)$ is true for all $k \in \mathbb{N}$, it is enough to show that

1. $P(1)$ is true, and

2. whenever $n \in \mathbb{N}$ and $P(n)$ is true, then $P(n + 1)$ is also true.

Exercise 2:   Which of the statements $P(k)$ listed above satisfy item 1? Which of them satisfy item 2? No proof is needed here. You are just building intuition.

In a proof by induction, showing that $P(1)$ is true is called the **base case**, whereas showing item 2 in Theorem 2.17 holds for $P(k)$ is called the **induction step**.

Schematically, here are the natural numbers:

•   •   •   •   •   •   •   •   •   •   ⋯

1

We will circle integer n if $P(n)$ is true. So showing $P(1)$ is true is showing that at least the natural number 1 has a circle around it:

$$\odot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots$$
$$1$$

There might be other circles, and we don't care about them (yet). We just need to know that the first natural number, 1, has a circle.

To show that $P(k)$ satisfies item 2 of Theorem 2.17 you need to show that every natural numbers that has a circle also has the property that the natural number to its right is also circled. How do you prove that all the elements of a mathematical collection have a given property? You consider an arbitrary member of the collection and prove that it has the desired property. If an arbitrary element of the collection does, then they all do.

The language for doing this is the same as we've used all along: you suppose that n is an arbitrary circled natural number. Then you prove, from this assumption and other known facts, that $n + 1$ must also be circled. More formally, you suppose that for some $n \in \mathbb{N}$ that $P(n)$ is true. You are assuming that the world looks like this:

$$\cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \quad \cdot \quad \odot \quad \cdot \quad \cdot \quad \cdot \quad \cdots$$
$$1 \qquad\qquad\qquad\qquad n$$

There may be other circles, but you don't know about them, and (for the most part) you don't really care about them either. The assumption that there is a circle about integer n is known as the **induction hypothesis**. It is important to realize you don't know if any natural numbers are circled. Maybe there are none.[1] You are simply showing that **if** one of them were, **then** the next one would be as well.
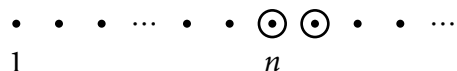
Here are the induction hypotheses you would use for the first few examples of statements $P(k)$ listed above.

- Suppose for some $n \in \mathbb{N}$ that $n \geq 1$.

- Suppose for some $n \in \mathbb{N}$ that $n^2 > n + 10$.

- Suppose for some $n \in \mathbb{N}$ that n is even.

Exercise 3: Write down the induction hypotheses for the remaining examples of statements $P(k)$.

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[1]If you proved the base case first, then you do know there is at least one circled natural number, namely 1. But most proofs by induction can prove the base case and the induction step in either order; its just traditional to prove the base case first. Within the proof of the induction step, you do not suppose and do not use the fact that 1 has a circle.

To complete the induction step, your job is to show that, that $P(n + 1)$ must also be true. That is, there must be at least one more circle:

$$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \odot \quad \odot \quad \bullet \quad \bullet \quad \cdots$$
$$1 \qquad\qquad\qquad\qquad\qquad n$$

Here's what needs to be proved to complete the induction step for the first few example statements.

- $n + 1 \geq 1$.

- $(n + 1)^2 > (n + 1) + 10$.

- $n + 1$ is even.

Exercise 4:   Finish this list for the remaining example statements $P(k)$.

In proving these facts, you are only allowed to assume that $P(n)$ is true, along with any known facts established before you started your proof by induction. Sometimes it can be awkward or tricky to show that $P(n + 1)$ is true only knowing that $P(n)$ is true. There is another version of induction that allows you to assume more.

Theorem 2.17'SI' (Strong Induction):   Suppose $P(k)$ is a statement depending on $k \in \mathbb{N}$. To show that $P(k)$ is true for all $k \in \mathbb{N}$, it is enough to show that

- whenever $n \in \mathbb{N}$ and $P(k)$ is true for all $k \in \mathbb{N}$ such that $k < n$, then $P(n)$ is also true.

We'll say that $P(k)$ satisfies the **strong induction condition** if it satisfies the bulleted item above. We'll prove this theorem in a moment. For now, let's see how this version of induction is different.

In order to show that $P(k)$ satisfies the strong induction condition, you consider an arbitrary $n \in \mathbb{N}$ such that $P(k)$ is true for all $k \in \mathbb{N}$ with $k < n$, and your job is to show that $P(n)$ is also true. So the inductive hypothesis in a proof via strong induction is that for some $n \in \mathbb{N}$ the world looks as follows:

$$\odot \quad \odot \quad \odot \quad \cdots \quad \odot \quad \odot \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \cdots$$
$$1 \qquad\qquad\qquad\qquad\quad n$$

That is, we get to assume that all the natural numbers to the left of n have a circle. Assuming all these facts, to complete the induction step we show that $P(n)$ must also be true, i.e.

$$\odot \; \odot \; \odot \; \cdots \; \odot \; \odot \; \odot \; \bullet \; \bullet \; \bullet \; \cdots$$
$$1 \hspace{4.5cm} n$$

You might be wondering what happened to the base case. To see why it isn't needed now, suppose that $P(k)$ is a statement that satisfies the strong induction condition. Is $P(1)$ is true? Since $P(k)$ satisfies the strong induction condition, we know that if $P(k)$ is true for all natural numbers with $k < 1$, then $P(1)$ will also be true. Look at the natural numbers:

$$\bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \cdots$$
$$1$$

Is it true that every natural number to the left of 1 has a circle? Yes! Every single natural number to the left of 1 has a circle. Not one of them fails to have a circle. We know that **if** $P(k)$ is true for all natural numbers less than 1, **then** $P(1)$ must be true. So the world must look at least like:

$$\odot \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \cdots$$
$$1$$
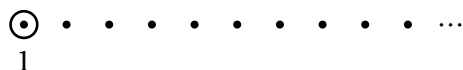
In conclusion: if $P(k)$ satisfies the strong induction condition, then $P(1)$ is necessarily true. So when you prove that $P(k)$ satisfies the strong induction condition, as a part of this demonstration you will necessarily be showing that $P(1)$ is true.

Here are some inductive hypotheses you would use for strong induction.

- Suppose for some $n \in \mathbb{N}$ that $k \geq 1$ for all $k \in \mathbb{N}$ such that $k < n$.

- Suppose for some $n \in \mathbb{N}$ that $k^2 > k + 10$ for all $k \in \mathbb{N}$ such that $k < n$.

- Suppose for some $n \in \mathbb{N}$ that k is even for all natural numbers $k < n$.

To complete the inductive step, you then prove the following:

- $n \geq 1$.

- $n^2 > n + 10$.

- n is even.

Exercise 5: For each of the remaining example statements $P(k)$ given on page 1, write down the strong inductive hypothesis, and what would have to be proved to complete the strong inductive step.

In doing a proof by strong induction, your inductive hypothesis will be that for some $n \in \mathbb{N}$ that $P(k)$ is true for every natural number less than n. Note that if n happened to be 1, (and it might be!) you wouldn't be assuming anything at all! In this way, your proof by strong induction will effectively prove the base case.

Here's an example of strong induction in action.

Theorem 2.32 (Well Ordering Principle):   Suppose $A \subseteq \mathbb{N}$ is non-empty. Then it has a smallest element.

Proof. Let $P(k)$ be the statement that "all subsets of $\mathbb{N}$ containing the integer k have a least element." If we can show that $P(k)$ is true for all $k \in \mathbb{N}$, then we will have shown that every nonempty subset of $\mathbb{N}$ has a least element. After all, suppose $A \subseteq \mathbb{N}$ is nonempty. Then there exists $k \in \mathbb{N}$ such that $k \in A$. Since $P(k)$ is true, A has a least element.

We show that $P(k)$ is true for all $k \in \mathbb{N}$ by strong induction.

Suppose for some $n \in \mathbb{N}$ that $P(k)$ is true for all natural numbers $k < n$. So we know that if k is a natural number less than n, and if a subset $A \subseteq \mathbb{N}$ contains k, then A has a minimal element. Our job is to show that $P(n)$ is true. That is, we need to show that if $A \subseteq \mathbb{N}$, and if $n \in A$, then A has a minimal element.

So let A be an arbitrary subset of $\mathbb{N}$ that contains n. Now either A contains a natural number less than n or it does not. If it does, our (strong) induction hypothesis implies that A has least element. Suppose A does not contain any natural numbers less than n. We will show that n is the least element of A, i.e. that

1. $n \in A$, and

2. $n \leq a$ whenever $a \in A$.

We have assumed that $n \in A$, so we only need to establish item 2.

Let $a \in A$. Then, $a \in \mathbb{N}$ as well. Since A does not contains any elements of $\mathbb{N}$ less than n, Proposition 2.8 implies that $a \geq n$. So n satisfies item 2 as well, and is hence the least element of A.   □

It remains to give the proof of Theorem 2.17'SI'.

Theorem 2.17'SI' (Strong Induction):   Suppose $P(k)$ is a statement depending on $k \in \mathbb{N}$. To show that $P(k)$ is true for all $k \in \mathbb{N}$, it is enough to show that that $P(k)$ satisfies the strong induction condition:

*   whenever $n \in \mathbb{N}$ and $P(k)$ is true for all $k \in \mathbb{N}$ such that $k < n$, then $P(n)$ is also true.

Proof. Suppose $P(k)$ is a statement satisfying the strong induction condition.

Let
$$T = \{n \in \mathbb{N} : P(k) \text{ is true for all natural numbers } k < n\}.$$

We will show that $T = \mathbb{N}$. Suppose for a moment that we have done this. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$ as well. Since $T = \mathbb{N}$, $n + 1 \in T$. This implies that $P(k)$ is true for all $k \in \mathbb{N}$ with $k < n + 1$. In particular, $P(n)$ is true. So if we can show that $T = \mathbb{N}$, then we can conclude that $P(k)$ is true for all $k \in \mathbb{N}$.

We will show that $T = \mathbb{N}$ by applying Proposition 2.16. That is, we will show that $1 \in \mathbb{N}$, and that whenever $n \in T$, then $n + 1 \in T$.

To see that $1 \in T$, we recall from Proposition 2.20 (and Proposition 2.8) that there are no natural numbers less than 1. Hence, for all $k \in \mathbb{N}$ such that $k < 1$, $P(k)$ is true. The strong induction condition then implies that $P(1)$ is true.

Suppose that for some $n \in \mathbb{N}$ that $n \in T$. So for all $k \in \mathbb{N}$ with $k < n$, $P(k)$ is true. We wish to show that $n + 1 \in T$, i.e. that $P(k)$ is true for all natural numbers k less than $n + 1$. Observe that $P(k)$ is true for all natural numbers $k < n$, since $n \in T$. Moreover, since $P(k)$ satisfies the strong induction condition, we know that $P(n)$ is true. Hence $P(k)$ is true for all natural numbers $k \leq n$. Now if $k \in \mathbb{N}$ and $k < n + 1$, Corollary 2.22 implies that $k \leq n$. Hence $P(k)$ is true for all $k \in \mathbb{N}$ such that $k < n + 1$. So $n + 1 \in T$, as desired. $\qquad\qquad\square$