Proposition 4.30: For all $k, m \in \mathbb{N}$, where $m \geq 2$,

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Proof. Let $m \in \mathbb{Z}_{\geq 2}$. We will show that

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$$

for all $k \in \mathbb{N}$ by strong induction.

Suppose for some $n \in \mathbb{N}$ that

$$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$$

for all $k \in \mathbb{N}$ such that $k < n$. We wish to show that

$$f_{m+n} = f_{m-1}f_n + f_m f_{n+1}$$

as well.

Now either $n = 1$, $n = 2$, or $n \geq 3$.

If $n = 1$,

$$f_{m+n} = f_{m+1} = f_{m-1} + f_m$$

since $m + 1 \geq 2 + 1 = 3$. But

$$\begin{aligned} f_{m-1} + f_m &= f_{m-1} \cdot 1 + f_m \cdot 1 \\ &= f_{m-1} \cdot f_1 + f_m \cdot f_2 \\ &= f_{m-1} \cdot f_n + f_m \cdot f_{n+1}. \end{aligned}$$

Hence

$$f_{m+n} = f_{m-1} \cdot f_n + f_m \cdot f_{n+1}.$$

If $n = 2$, since $m + 2 \geq 3$ and $m + 1 \geq 3$,

$$f_{m+n} = f_{m+2} = f_m + f_{m+1} = f_m + f_{m-1} + f_m = f_{m-1} \cdot 1 + f_m \cdot 2.$$

But

$$\begin{aligned} f_{m-1} \cdot 1 + f_m \cdot 2 &= f_{m-1} \cdot f_2 + f_m \cdot f_3 \\ &= f_{m-1} \cdot f_n + f_m \cdot f_{n+1}. \end{aligned}$$

So

$$f_{m+n} = f_{m-1} \cdot f_n + f_m \cdot f_{n+1}.$$

Suppose $n \geq 3$. Since $m + n \geq 3$,

$$f_{m+n} = f_{m+n-2} + f_{m+n-1}.$$

Since $n \geq 3$, $1 \leq n-1 < n$ and $1 \leq n-2 < n$, and hence, by the inductive hypothesis,

$$\begin{aligned} f_{m+n-2} &= f_{m-1}f_{n-2} + f_m f_{n-1} \quad \text{and} \\ f_{m+n-1} &= f_{m-1}f_{n-1} + f_m f_n. \end{aligned}$$

So

$$\begin{aligned} f_{m+n} &= (f_{m-1}f_{n-2} + f_m f_{n-1}) + (f_{m-1}f_{n-1} + f_m f_n) \\ &= f_{m-1}\left[f_{n-2} + f_{n-1}\right] + f_m\left[f_{n-1} + f_n\right]. \end{aligned}$$

Since $n \geq 3$, $f_n = f_{n-2} + f_{n-1}$ and $f_{n+1} = f_{n-1} + f_n$. So

$$f_{m+n} = f_{m-1}f_n + f_m f_{n+1}.$$

□

An integer n is **odd** if there exists an integer j such that $n = 2j + 1$.

Proposition 9.A: Every integer is either even or odd, and no integer is both.

Proof. We first show that if $n \in \mathbb{Z}_{\geq 0}$ that n is either even or odd by induction on n. The base case is clear because $0 = 2 \cdot 0$ is even. Suppose for some $n \in \mathbb{Z}_{\geq 0}$ that n is either even or odd. Suppose n is even, so $n = 2j$ for some $j \in \mathbb{Z}$. Then $n+1 = 2j+1$ and $n+1$ is odd. Suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. So $n+1 = 2k+2 = 2(k+1)$. So $n+1$ is even. Thus, regardless of whether n is even or odd, $n+1$ is either even or odd.

We have shown that non-negative integers are either even or odd. We now show the same holds for negative integers as well. Suppose $x \in \mathbb{Z}$ and $x < 0$. Let $y = -x$, so $y > 0$ and y is either even or odd. Suppose y is even, so $y = 2j$ for some $j \in \mathbb{Z}$. Then $x = -y = 2(-j)$, so x is even. Suppose y is odd, so $y = 2k$ for some $k \in \mathbb{Z}$. Then

$$x = -y = 2(-k) - 1 = 2(-k) - 2 + 1 = 2(-k-1) + 1.$$

So x is odd. Thus x is either even or odd.

Since every integer is either non-negative or negative, we have shown that all integers are either even or odd.

Now suppose to produce a contradiction that $x \in \mathbb{Z}$ and x is both even and odd. So $x = 2k + 1$ and $x = 2j$ for certain integers j and k. Hence

$$2k + 1 = 2j$$

and

$$2(j - k) = 1.$$

Hence $2 \mid 1$. Since $2 > 1$, this contradicts Proposition 2.23. □

Proposition 6.5: Assume we are given an equivalence relation on a set A. For all $a_1, a_2 \in A$, either $[a_1] = [a_2]$ or $[a_1] \cap [a_2] = \emptyset$.

Proof. Let $a_1, a_2 \in A$. It suffices to show that if $[a_1] \cap [a_2] \neq \emptyset$ then $[a_1] = [a_2]$. Suppose $b \in [a_1] \cap [a_2]$. Then $a_1 \sim b$ and $a_2 \sim b$ by the definition of equivalence classes. By transitivity, $a_1 \sim a_2$. So by Propositoin 6.4(ii), $[a_1] = [a_2]$. □

Proposition 6.6 (Partial)**:** Let A be a set and let Π be a partition of A. We define $a \sim b$ if there exists $P \in \Pi$ such that $a \in P$ and $b \in P$. Then $\sim$ is an equivalence relation.

Proof. We must show that $\sim$ is reflexive, symmetric, and transitive.

bf Reflexive: Suppose $a \in A$. Since Π is a partition, there exists $P \in \Pi$ such that $a \in P$. Hence $a \in P$ and $a \in P$, so $a \sim a$.

Symmetric: Suppose $a, b \in A$ and $a \sim b$. Then there exists $P \in \Pi$ such that $a, b \in P$. Since $b, a \in P$, $b \sim a$.

Transitive: Suppose $a, b, c \in A$ and $a \sim b$ and $b \sim c$. Then there exists $P \in \Pi$ such that $a, b \in P$ and there exists $Q \in \Pi$ such that $b, c \in Q$. Note that $b \in P$ and $b \in Q$. Since Π is a partition, and since $P \cap Q \neq \emptyset$, it follows that $P = Q$. Since $a \in P$ and $c \in Q = P$, $a \sim c$. □

Project 6.7: For each of the following relations defined on $\mathbb{Z}$, determine whether it is an equivalence relation. If it is, determine its equivalence classes.

1. $x \sim y$ if $x < y$. This is not an equivalence relation. Note that $0 \nsim 0$, so $\sim$ is not reflexive.

2. $x \sim y$ if $x \leq y$. This is not an equivalence relation. Note that $0 \sim 1$ but $1 \nsim 0$, so $\sim$ is not symmetric.

3. $x \sim y$ if $|x| = |y|$. This is an equivalence relation. If $x \in \mathbb{Z}$, clearly $|x| = |x|$ and hence $x \sim x$. Suppose $x, y \in \mathbb{Z}$ and $x \sim y$. Then $|x| = |y|$. Since $|y| = |x|$, $y \sim x$. Suppose $x, y, z \in \mathbb{Z}$, $x \sim y$, and $y \sim z$. Then $|x| = |y|$ and $|y| = |z|$ so $|x| = |z|$. Hence $x \sim z$.

   If $x \in \mathbb{Z}$, $[x] = \{x, -x\}$.

4. $x \sim y$ if $x \neq y$. This is not an equivalence relation. Since $0 \nsim 0$, it is not reflexive.

5. $x \sim y$ if $xy > 0$. This is not an equivalence relation. Note that $0 \nsim 0$ as $0 \cdot 0 = 0$, so $0 \cdot 0 \not> 0$. Hence $\sim$ is not reflexive.

6. $x \sim y$ if $x \mid y$ or $y \mid x$. This is not an equivalence relation. Note that $2 \mid 10$ and $5 \mid 10$, so $2 \sim 10$ and $10 \sim 5$. But $2 \nmid 5$ and $5 \nmid 2$ so $2 \nsim 5$ and transitivity fails.

Proposition 6.17: Let $m \in \mathbb{Z}$. Then m is even if and only if m^2 is even.

Proof. Suppose m is even. Then there exists $j \in \mathbb{Z}$ such that $m = 2j$. Since $m^2 = 2(2j^2)$, m^2 is even.

We prove the converse using the contrapositive. Suppose m is not even. By Proposition 6.15, there exists $j \in \mathbb{Z}$ such that $m = 2j + 1$. Hence

$$m^2 = (2j+1)(2j+1) = 2(j(2j+1)+j)+1.$$

By Proposition 6.14, m^2 is odd, so m^2 is not even. □