Proposition 4.5: For all $k \in \mathbb{Z}_{\geq 0}$, $k! \in \mathbb{N}$.

Proof. We proceed by induction on $k \geq 0$. When $k = 0$, $k! = 0! = 1 \in \mathbb{N}$. This establishes the base case. Suppose for some $k \in \mathbb{Z}_{\geq 0}$ that $k! \in \mathbb{N}$. Since $k \geq 0$, $k + 1 \geq 1 \in \mathbb{N}$. Since $k! \in \mathbb{N}$ as well,

$$(k + 1)! = k!(k + 1) \in \mathbb{N}.$$

□

Proposition 4.7 (i): For all $k \in \mathbb{N}$, $5^{2k} - 1$ is divisible by 24.

Proof. We proceed by induction on $k \in \mathbb{N}$. Observe that when $k = 0$, $5^{2k} - 1 = 5^0 - 1 = 0$. Since $0 \mid 0$, the base case is established.

Suppose for some $k \in \mathbb{N}$ that $24 \mid 5^{2k} - 1$. Hence there exists an integer j such that $5^{2k} - 1 = 24j$. Then

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k}5^2 - 1 \\ &= (5^{2k} - 1)5^2 + 5^2 - 1 \\ &= 24j \cdot 25 + 24 \\ &= 24(25j + 1). \end{aligned}$$

Hence $24 \mid 5^{2(k+1)} - 1$. □

Proposition 4.6(iii): Let $b \in \mathbb{Z}$ and $m, k \geq 0$. Then $(b^m)^k = b^{mk}$.

Proof. Let $m \in \mathbb{Z}_{\geq 0}$. We will prove that $(b^m)^k = b^{mk}$ for all $k \in \mathbb{Z}_{\geq 0}$ by induction.

When $k = 0$,

$$(b^m)^k = (b^m)^0 = 1 = b^0 = b^{m0} = b^{mk}.$$

Suppose for some $k \in \mathbb{Z}_{\geq 0}$ that $(b^m)^k = b^{mk}$. Then

$$\begin{aligned} (b^m)^{k+1} &= (b^m)^k(b^m) \\ &= b^{mk}b^m && \text{(by the induction hypothesis)} \\ &= b^{mk+m} && \text{(by Proposition 4.6(ii))} \\ &= b^{m(k+1)}. \end{aligned}$$

□

Proposition 4.11: For all $k \in \mathbb{N}$,

$$2\sum_{j=1}^{k} j = k(k + 1).$$

Proof. We proceed by induction on $k \in \mathbb{N}$. When $k = 1$,

$$2\sum_{j=1}^{k} j = 2\sum_{j=1}^{1} j = 2 \cdot 1 = 1 \cdot (1+1) = k \cdot (k+1).$$

Suppose for some $k \in \mathbb{N}$ that

$$2\sum_{j=1}^{k} j = k \cdot (k+1).$$

Then

$$\begin{aligned}
2\sum_{j=1}^{k+1} j &= 2\left[\sum_{j=1}^{k} j + (k+1)\right] \\
&= 2\sum_{j=1}^{k} j + 2(k+1) \\
&= k(k+1) + 2(k+1) && \text{(by the induction hypothesis)} \\
&= (k+1)(k+2) \\
&= (k+1)((k+1)+1). && \square
\end{aligned}$$

Proposition 4.A: Suppose a and b are integers such that $a \neq 0$ and $a \mid b$. Then there exists a unique integer j such that $b = aj$.

Proof. Suppose a and b are integers such that $a \mid b$. Then there exists $j \in \mathbb{Z}$ such that $b = aj$. Suppose for some $k \in \mathbb{Z}$ that $b = ak$. Then

$$aj = ak$$

and, since $a \neq 0$, Axiom 2.1 implies that $k = j$. Thus there is a unique integer j such that $b = aj$. □

Proposition 4.8: For all $k \in \mathbb{N}$, $4^k > k$.

Proof. We proceed by induction on $k \in \mathbb{N}$. When $k = 1$,

$$4^k = 4^1 = 4 > 1 = k.$$

Suppose for some $k \in \mathbb{N}$ that $4^k > 1$. Then

$$\begin{aligned}
4^{k+1} &= 4^k 4 \\
&> k \cdot 4 && \text{(by the induction hypothesis and Proposition 4.D)} \\
&= k + 3k \\
&\geq k + 1
\end{aligned}$$

since $3k \geq 3 \cdot 1 = 3 > 1$. Hence $4^{k+1} \geq k + 1$ as required. □

Proposition 4.13: For $x \neq 1$ and $k \in \mathbb{Z}_{\geq 0}$, $\displaystyle\sum_{j=0}^{k} x^j = \frac{1 - x^{k+1}}{1 - x}$.

Hint: Show that $(1 - x) \sum_{j=0}^{k} x^j = 1 - x^{k+1}$.

Proof. Since $x \neq 1$, the statement

$$\sum_{j=0}^{k} x^j = \frac{1 - x^{k+1}}{1 - x}$$

is equivalent to

$$(1 - x) \sum_{j=0}^{k} x^j = 1 - x^{k+1}.$$

We will show that the latter statement holds for all $k \in \mathbb{Z}_{\geq 0}$ by induction.

When $k = 0$,

$$(1 - x) \sum_{j=0}^{k} x^j = (1 - x) \sum_{j=0}^{0} x^j = (1 - x)x^0 = 1 - x = 1 - x^1 = 1 - x^{k+1}.$$

Suppose for some $k \in \mathbb{Z}_{\geq 0}$ that

$$(1 - x) \sum_{j=0}^{k} x^j = 1 - x^{k+1}.$$

Then

$$\begin{aligned}
(1 - x) \sum_{j=0}^{k+1} x^j &= (1 - x) \sum_{j=0}^{k} x^j + (1 - x)x^{k+1} \\
&= 1 - x^{k+1} + (1 - x)x^{k+1} && \text{(by the induction hypothesis)} \\
&= 1 - x^{k+2} \\
&= 1 - x^{(k+1)+1}. && \square
\end{aligned}$$

Proposition 4.15(i): Let $m \in \mathbb{Z}$ and $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$. If then for all $k \in \mathbb{N}$

$$\sum_{j=1}^{k} m x_j = m \sum_{j=1}^{k} x_j.$$

Proof. We proceed by induction on $k \in \mathbb{N}$. When $k = 1$,

$$\sum_{j=1}^{k} m x_j = \sum_{j=1}^{1} m x_j = m x_1 = m \sum_{j=1}^{1} x_j = m \sum_{j=1}^{k} x_j.$$

Suppose for some $k \in \mathbb{N}$ that

$$\sum_{j=1}^{k} mx_j = m \sum_{j=1}^{k} x_j.$$

Then

$$\begin{aligned}
\sum_{j=1}^{k+1} mx_j &= \sum_{j=1}^{k} mx_j + mx_{k+1} \\
&= m \sum_{j=1}^{k} x_j + mx_{k+1} && \text{(by the induction hypothesis)} \\
&= m \left[\sum_{j=1}^{k} x_j + x_{k+1} \right] \\
&= m \sum_{j=1}^{k+1} x_j. && \square
\end{aligned}$$

Proposition 4.15(iii): Let $(x_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{Z}$. If $x_j = n \in \mathbb{Z}$ for all $j \in \mathbb{N}$ then for all $k \in \mathbb{N}$

$$\sum_{j=1}^{k} x_j = kn.$$

Proof. We proceed by induction on $k \in \mathbb{N}$. When $k = 1$,

$$\sum_{j=1}^{k} x_j = \sum_{j=1}^{1} x_j = x_1 = n = 1 \cdot n = kn.$$

Suppose for some $k \in \mathbb{N}$ that $\sum_{j=1}^{k} x_j = kn$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} x_j &= \sum_{j=1}^{k} x_j + x_{k+1} \\
&= \sum_{j=1}^{k} x_j + x_{k+1} \\
&= kn + x_{k+1} && \text{(by the induction hypothesis)} \\
&= kn + n \\
&= (k+1)n. && \square
\end{aligned}$$

Proposition 4.16(ii): Let $(x_j)_{j=m}^{\infty}$ and $(y_j)_{j=m}^{\infty}$ be sequences in $\mathbb{Z}$. For all $a, b \in \mathbb{Z}$ such that $m \leq a \leq b$,

$$\sum_{j=a}^{b} (x_j + y_j) = \sum_{j=a}^{b} x_j + \sum_{j=a}^{b} y_j.$$

Proof. Let $a \in \mathbb{Z}_{\geq m}$. We will show that

$$\sum_{j=a}^{b}(x_j + y_j) = \sum_{j=a}^{b} x_j + \sum_{j=a}^{b} y_j.$$

for all $b \geq a$ by induction. When $b = a$,

$$\sum_{j=a}^{b}(x_j + y_j) = \sum_{j=a}^{a}(x_j + y_j) = x_a + y_a = \sum_{j=a}^{a} x_j + \sum_{j=a}^{a} y_j = \sum_{j=a}^{b} x_j + \sum_{j=a}^{b} y_j.$$

Suppose for some $b \geq a$ that

$$\sum_{j=a}^{b}(x_j + y_j) = \sum_{j=a}^{b} x_j + \sum_{j=a}^{b} y_j.$$

Then

$$\begin{aligned}
\sum_{j=a}^{b+1}(x_j + y_j) &= \sum_{j=a}^{b}(x_j + y_j) + x_{b+1} + y_{b+1} \\
&= \sum_{j=a}^{b} x_j + \sum_{j=a}^{b} y_j + x_{b+1} + y_{b+1} \qquad \text{(by the induction hypothesis)} \\
&= \sum_{j=a}^{b} x_j + x_{b+1} \sum_{j=a}^{b} y_j + y_{b+1} \\
&= \sum_{j=a}^{b+1} x_j + \sum_{j=a}^{b+1} y_j. \qquad \square
\end{aligned}$$

Proposition 4.18: Let $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$ such that $x_j \leq y_j$ for all $j \in \mathbb{N}$. Then for all $k \in \mathbb{N}$,

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j.$$

Proof. We proceed by induction on $k \geq 1$. When $k = 1$,

$$\sum_{j=1}^{k} x_j = \sum_{j=1}^{1} x_j = x_1 \leq y_1 = \sum_{j=1}^{1} y_j = \sum_{j=1}^{k} y_j.$$

Suppose for some $k \geq 1$ that

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j.$$

Then

$$
\begin{aligned}
\sum_{j=1}^{k+1} x_j &= \sum_{j=1}^{k} x_j + x_{k+1} \\
&\leq \sum_{j=1}^{k} y_j + x_{k+1} && \text{(by the induction hypothesis)} \\
&\leq sum_{j=1}^{k} y_j + y_{k+1} \\
&= sum_{j=1}^{k+1} y_j. && \square
\end{aligned}
$$