Proposition 2.33: Let A be a nonempty subset of $\mathbb{Z}$. Suppose for some $b \in \mathbb{Z}$ that $b \leq a$ for all $a \in A$. Then A has a least element.

Proof . Let A be a nonempty subset of $\mathbb{Z}$ that admits a lower bound $b \in \mathbb{Z}$. So $b \leq a$ for all $a \in A$. Let $m = -b + 1$ and define

$$C = \{a + m : a \in A\}.$$

We claim that C is a nonempty subset of $\mathbb{N}$.

To see that it is nonempty, let $a \in A$. We know such an a exists since A is nonempty. Then $a + m \in C$, so C is nonempty.

We now wish to show that $C \subseteq \mathbb{N}$. Let $c \in C$. Then there exists $a \in A$ such that $c = a + m$. Since $a \in A$, $a \geq b$. So

$$c = a + m \geq b + m = b + 1 - b = 1.$$

So $c \geq 1$ and $c \in \mathbb{N}$. Hence $C \subseteq \mathbb{N}$.

Since C is a nonempty collection of natural numbers, it admits a least element c_0. Since $c_0 \in C$, there exists $a_0 \in A$ such that $c_0 = a_0 + m$. We claim that a_0 is the least element of A. Since $a_0 \in A$, we need only show that for all $a \in A$, $a_0 \leq a$.

Let $a \in A$. Then $a + m \in C$, so $c_0 \leq a + m$. Hence $a_0 + m \leq a + m$ and consequently $a_0 \leq a$ as desired. □

Proposition HW7.A: Let $m, n, p \in \mathbb{Z}$ and suppose $p > 0$. If $mp \leq np$ then $m \leq n$.

Proof . Let $m, n, p \in \mathbb{Z}$ and suppose $p > 0$. We will show that if $m > n$ then $mp > np$, which is the contrapositive of the desired implication. Suppose $m > n$. Then, since $p > 0$, Proposition 2.C implies $mp > np$. □

Proposition 5.4: Let A, B, C be sets.

(i) $A = A$.

(ii) If $A = B$ then $B = A$.

(iii) If $A = B$ and $B = C$ then $A = C$.

Proof. Proof of i): By Proposition 5.1 i), $A \subseteq A$ and $A \subseteq A$, so $A = A$.

Proof of ii): Suppose $A = B$. Then $A \subseteq B$ and $B \subseteq A$ by definition. But then $B \subseteq A$ and $A \subseteq B$, so $B = A$.

Proof of (iii): Suppose $A = B$ and $B = C$ Then $A \subseteq B$ and $B \subseteq C$. Proposition 5.1(ii) then implies $A \subseteq C$. Similarly, since $C \subseteq B$ and $B \subseteq A$, $C \subseteq A$. Hence $A = C$. □

Project 5.12 (partial): For each of the following double implications $P \iff Q$ determine which of the implications $P \implies Q$ or $Q \implies P$, if any, are true. For the ones that are true, prove them. For the ones are are not true, provide a counterexample.

(ii) $C \subseteq A$ or $C \subseteq B \iff C \subseteq (A \cup B)$

(iii) $C \subseteq A$ and $C \subseteq B \iff C \subseteq (A \cap B)$

It is true that if $C \subseteq A$ or $C \subseteq B$ then $C \subseteq (A \cup B)$, but the converse is false.

To see that the converse is false, consider $A = \{1\}$, $B = \{2\}$ and $C = \{1, 2\}$. Then $C \subseteq A \cup B$ But $C \nsubseteq A$ and $C \nsubseteq B$.

We now show that the remaining implications are true.

Proof. Let A, B, and C be sets.

Suppose $C \subseteq A$ or $C \subseteq B$. Let $c \in C$. Then either $c \in A$ or $c \in B$. So $c \in A \cup B$. Hence $C \subseteq A \cup B$.

Suppose $C \subseteq A$ and $C \subseteq B$. Let $c \in C$. Then $c \in A$ and $c \in B$. Hence $c \in A \cap B$. So $C \subseteq A \cap B$.

Suppose $C \subseteq A \cap B$. Let $c \in C$. Then $c \in A \cap B$. Hence $c \in A$ and $c \in B$. Since c is arbitrary, $C \subseteq A$ and $C \subseteq B$. □

Proposition 5.15 (DeMorgan's Laws): Given two subsets $A, B \subseteq X$,

(i) $(A \cap B)^c = A^c \cup B^c$

(ii) $(A \cup B)^c = A^c \cap B^c$

Proof. We prove part (ii) first. Suppose $x \in (A \cup B)^c$. Then $x \in X$ and $x \notin (A \cup B)$. Since $x \notin A \cup B$, it is not true that $x \in A$ or $x \in B$. Hence $x \notin A$ and $x \notin B$. Since $x \in X$ we conclude that $x \in A^c$ and $x \in B^c$. So $x \in A^c \cup B^c$. Hence $(A \cup B)^c \subseteq A^c \cap B^c$.

Suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. So $x \in X$, $x \notin A$, and $x \notin B$. Hence $x \notin A \cup B$. Since $x \in X$, we conclude that $x \in (A \cup B)^c$. So $(A \cup B)^c \supseteq A^c \cap B^c$ and therefore $(A \cup B)^c = A^c \cap B^c$.

To prove part (i), we now apply part (ii) to A^c and B^c, so

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c.$$

By Corollary 5.14b, $(A^c)^c = A$ and $(B^c)^c = B$. So

$$(A^c \cup B^c)^c = A \cap B.$$

Taking the complement of both sides, and applying Corollary 5.14 again to the left-hand side we conclude that

$$A^c \cup B^c = (A \cap B)^c.$$

□

Proposition 5.20: Let A, B, and C be sets.

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof. (i) Suppose $x \in A \times (B \cup C)$. Then there exists $a \in A$ and $d \in (B \cup C)$ such that $x = (a, d)$. Since $d \in (B \cup C)$, it follows that $d \in B$ and $d \in C$. Since $a \in A$ and $d \in B$, $z = (a, d) \in A \times B$. Since $a \in A$ and $d \in C$, $z = (a, d) \in A \times C$. Hence $z \in (A \times B) \cap (A \times C)$. Therefore $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Suppose $x \in\in (A \times B) \cap (A \times C)$. Then $x = (a, d)$ for some $a \in A$ and where $d \in B$ and $d \in C$. Note that $d \in B \cap C$ and hence $x \in A \times (B \cap C)$. Hence $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ as well, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) Suppose $z \in A \times (B \cap C)$. Then there exists $a \in A$ and $d \in B \cap C$ such that $z = (a, d)$. Since $d \in B \cap C$, $d \in B$ and $d \in C$. Since $a \in A$ and $d \in B$, $z = (a, d) \in (A, B)$. Since $a \in A$ and $d \in C$, $z = (a, d) \in (A, C)$. Consequently $z \in (A \times B) \cap (A \times C)$ and $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

Suppose $z \in (A \times B) \cap (A \times C)$. Then $z \in (A \times B)$ and $z \in (A \times C)$, so $z = (a, d)$ for some $a \in A$ and d such that $d \in B$ and $d \in C$. But then $d \in B \cap C$ and hence $z = (a, d) \in A \times (B \cap C)$. So $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Since $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$, $A \times (B \cap C) = (A \times B) \cap (A \times C)$. □