Project 2.35: Compute gcd(4, 6), gcd(7, 13), gcd(−4, 10) and gcd(−5, −15). You do **NOT** have to prove that you have found the gcd. But you do have to exhibit the integers x and y in the definition of the gcd.

Solution:
Observe

$$\gcd(4,6) = 2 = 4 \cdot (-1) + 6 \cdot 1.$$

Other elements of $S_{4,6}$ include

$$\begin{aligned} 4 &= 4 \cdot 1 + 6 \cdot 0 \\ 6 &= 4 \cdot 0 + 6 \cdot 1 \\ 14 &= 4 \cdot (-1) + 6 \cdot 3. \end{aligned}$$

Observe

$$\gcd(7,13) = 1 = 7 \cdot 2 + 13 \cdot (-1).$$

Other elements of $S_{7,13}$ include

$$\begin{aligned} 2 &= 7 \cdot 2 + 13 \cdot (-2) \\ 3 &= 7 \cdot 3 + 13 \cdot (-3) \\ 4 &= 7 \cdot 4 + 13 \cdot (-4). \end{aligned}$$

Observe

$$\gcd(-4,10) = 2 = (-4) \cdot 2 + 10 \cdot 1.$$

Other elements of $S_{-4,10}$ include

$$\begin{aligned} 4 &= (-4) \cdot 4 + 10 \cdot 2 \\ 6 &= (-4) \cdot 6 + 10 \cdot 3 \\ 8 &= (-4) \cdot 8 + 10 \cdot 4. \end{aligned}$$

Finally, observe

$$\gcd(-5,15) = 5 = (-5) \cdot 2 + 15 \cdot (1).$$

Other elements of $S_{-5,15}$ include

$$\begin{aligned} 10 &= (-5) \cdot 4 + 15 \cdot 2 \\ 20 &= (-5) \cdot 6 + 15 \cdot 3 \\ 30 &= (-5) \cdot 8 + 15 \cdot 4. \end{aligned}$$

Project 3.1: Express each of the following statements using quantifiers.

(i) There exists a smallest natural number.

(ii) There does not exist a smallest integer.

(iii) Every integer is the product of two integers.

(iv) The equation $x^2 - 2y^2 = 3$ has an integer solution.

Solution:

(i) $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N}$ such that$)$ $n \leq m$

(ii) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N}$ such that$)$ $m < n$

(iii) $(\forall n \in \mathbb{N})(\exists m, p \in \mathbb{N}$ such that$)$ $n = mp$

(iv) $(\exists x, y, \in \mathbb{Z}$ such that$)$ $x^2 - 2y^2 = 3$.

Project 3.7: Negate each of the following statements

(i) Every cubic polynomial has a real root.

(ii) G is normal and H is regular.

(iii) $\exists! 0$ such that $\forall x$, $x + 0 = x$

(iv) The newspaper article was neither accurate nor entertaining.

(v) If $\gcd(m, n)$ is odd then m or n is odd.

(vi) H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G

(vii) For each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$

Solution:

(i) There exists a cubic polynomial that does not have a real root.

(ii) Either G is not normal or H is not regular.

(iii) Either there is no integer 0 such that $x + 0 = x$ for all $x \in \mathbb{Z}$, or there exist two distinct integers 0_1 and 0_2 such that $x + 0_1 = x$ and $x + 0_2 = x$ for all $x \in \mathbb{Z}$.

(iv) The newspaper article was either accurate or entertaining.

(v) There exist even integers m, n such that $\gcd(m, n)$ is odd.

(vi) Either there exists H, G, and N such that H/N is a normal subgroup of G/N but H is not a normal subgroup of G, or there exists H, G, and N such that H is a normal subgroup of G but H/N is not a normal subgroup of G/N.

(vii) There exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ and $|a_n - L| \geq \epsilon$.