Proposition HW5.1: The integer 1 is not divisible by 2. That is, $2 \nmid 1$.

Proof. Suppose to produce a contradiction that $2 \mid 1$. Then there is an integer j such that $1 = 2j$. Since $1, 2 \in \mathbb{N}$, Proposition 2.11 implies $j \in \mathbb{N}$. Since $0 < 1 < 2$ and $0 < 1 \leq j$, Proposition 2.7(iii) implies

$$1 \cdot 1 < 2j.$$

That is, $1 < 2j$. Since $1 = 2j$ as well, we have a contradiction of Proposition 2.8. □

Proposition HW5.2: Let $A = \{3x - 1 : x \in \mathbb{Z}\}$ and let $B = \{3x + 8 : x \in \mathbb{Z}\}$. Then $A = B$.

Proof. Suppose $a \in A$. Then there is an integer x such that $a = 3x - 1$. Let $y = x - 3$, so $x = y + 3$. Then

$$a = 3x - 1 = 3(y + 3) - 1 = 3y + 8.$$

Since $y \in \mathbb{Z}$ we conclude that $a \in B$. So $A \subseteq B$.

Suppose $b \in B$. Then there is an integer y such that $b = 3y + 8$. Let $x = y + 3$, so $y = x - 3$. Then

$$b = 3y + 8 = 3(x - 3) + 8 = 3x - 1.$$

Since $x \in \mathbb{Z}$ we conclude that $b \in A$. So $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, $A = B$. □

Proposition 2.21: There are no integers x such that $0 < x < 1$.

Proof. Suppose to the contrary that $x \in \mathbb{Z}$ and $0 < x < 1$. Since $x > 0$, $x \in \mathbb{N}$. By Proposition 2.20, $x \geq 1$. Since $x < 1$ as well, we have a contradiction of Proposition 2.8. □

Corollary 2.22: Let $n \in \mathbb{Z}$. There are no integers x such that $n < x < n + 1$.

Proof. Let $n \in \mathbb{N}$. Suppose to the contrary that there exists $x \in \mathbb{Z}$ such that $n < x < n + 1$. Since $n < x$, Proposition 2.7(i) implies

$$0 < x - n.$$

Since $x < n + 1$, Proposition 2.7(i) implies that

$$x - n < 1.$$

So

$$0 < x - n < 1.$$

Since $x - n \in \mathbb{Z}$, this is a contradiction of Proposition 2.21. □

Proposition 2.23: Let $m, n \in \mathbb{N}$. If n is divisible by m, then $m \leq n$.

Proof. Let $m, n \in \mathbb{N}$ such that $m \mid n$. Since $m \mid n$, there is an integer j such that

$$n = jm.$$

Since $n \in \mathbb{N}$ and $m \in \mathbb{N}$, Proposition 2.11 implies that $j \in \mathbb{N}$. Proposition 2.20 then implies that $j \geq 1$. Since $m \in \mathbb{N}$, $m > 0$. So we may apply Proposition 2.D to the inequality $j \geq 1$ to find $mj \geq 1 \cdot m = m$. Since $n = mj$, we conclude that $n \geq m$. □

Proposition 2.24: For all $k \in \mathbb{N}$, $k^2 + 1 > k$.

Proof. We proceed by induction on k. When $k = 1$, $k^2 + 1 = 2 > 1 = k$. Suppose for some $n \in \mathbb{N}$ that $n^2 + 1 > n$. Then

$$\begin{aligned}(n+1)^2 + 1 &= n^2 + 2n + 2 \\ &> n^2 + 2 && \text{by Prop. 2.7(i), since } 2n > 0 \\ &= n^2 + 1 + 1 \\ &> n + 1 && \text{since } n^2 + 1 > n.\end{aligned}$$

□

Proposition 2.27: For all $k \in \mathbb{Z}$ such that $k \geq 2$, $k^2 < k^3$.

Proof. We proceed by induction on $k \geq 2$. When $k = 2$, $k^2 = 4 < 8 = k^3$. Suppose for some $n \in \mathbb{N}$ that $n^2 < n^3$. Then

$$\begin{aligned}(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\ &= (n^2 + 2n + 1) + n^3 + n^2 + n \\ &= (n+1)^2 + n^3 + n^2 + n.\end{aligned}$$

Since $n \in \mathbb{N}$, Axiom 2.1(ii) implies that $n^2 \in \mathbb{N}$. Since $n \in \mathbb{N}$, Axiom 2.1(ii) implies that $n^3 = n^2 \cdot n \in \mathbb{N}$. Hence $n^3 + n^2 + n \in \mathbb{N}$ by two applications of Axiom 2.1(ii). So

$$(n+1)^3 = (n+1)^2 + n^3 + n^2 + n > (n+1)^2$$

by Proposition 2.7(i). □

Notice that the induction step in the proof above never used the induction hypothesis! This is a hint that we don't need to use induction. Here's a non-inductive proof.

Proof. Let $k \in \mathbb{Z}$ such that $k \geq 2$. Then $k \geq 2 > 1 > 0$. In particular, $k^2 > 0$ since $k \in \mathbb{N}$. Now $0 < 1 < k$ and $0 < k^2$ so Proposition 2.7(iii) implies

$$k^3 = k^2 \cdot k > k^2 \cdot 1 = k^2.$$

□