Proposition 2.E: Suppose $n, m \in \mathbb{Z}$ and $n > m$. Then $-n < -m$.

Proof. Let $n, m \in \mathbb{Z}$ such that $n > m$. So $n - m \in \mathbb{N}$. Observe that

$$\begin{aligned} n - m &= n + (-m) && \text{(definition of subtraction)} \\ &= -m + n \\ &= -m + (-(-n)) && \text{(Proposition 1.22(i))} \\ &= -m - (-n) && \text{(definition of subtraction).} \end{aligned}$$

Hence $-m - (-n) \in \mathbb{N}$ and $-m > -n$. □

Proposition 2.G: Suppose $n, m, p \in \mathbb{Z}$, $n > m$, and $p < 0$. Then $pn < pm$.

Proof. Suppose $n, n, p \in \mathbb{Z}$ such that $n > m$ and $p < 0$. By Proposition 2.E, $-p > -0 = 0$. Hence, by Proposition 2.C,

$$(-p)n > (-p)m$$

and therefore, by Proposition 1.25(iii)

$$-(pn) > -(pm).$$

Applying Proposition 2.E again we find

$$-(-pn) < -(-pm)$$

and therefore, by Proposition 1.22(i)

$$pn < pm.$$

□

Lemma 2.10a: $-1 < 0$

Proof. Notice that $0 - (-1) = 1 \in \mathbb{N}$. □

Proposition 2.10: The equation $x^2 = -1$ has no solution in $\mathbb{Z}$.

Proof. We will prove that the contrapositive is true, i.e. that if $x \in \mathbb{Z}$, then $x^2 \neq -1$. Let $x \in \mathbb{Z}$. Now either $x = 0$ or $x \neq 0$. If $x = 0$ then $x^2 = 0 \cdot 0 = 0$. Since $-1 < 0$, Proposition 2.8 implies that $-1 \neq 0$. So $x^2 \neq -1$.

On the other hand, suppose that $x \neq 0$. Then Proposition 2.9 implies that $x^2 \in \mathbb{N}$. That is, $x^2 > 0$. Since $-1 < 0$, Proposition 2.8 implies $m^2 \neq -1$. □

Proposition 2.11: Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. If $mn \in \mathbb{N}$ then $n \in \mathbb{N}$.

We could prove this result either using the contrapositive or by contradiction. I've provided both versions below. I think the proof using the contrapositive is cleaner. Some proofs by contradiction can only be done by contradiction. In other cases they are really proofs by the contrapositive in disguise!

Proof. (Contrapositive version) We will show that if $m \in \mathbb{N}$ and $n \notin \mathbb{N}$, then $mn \notin \mathbb{N}$.

Suppose $m \in \mathbb{N}$ and $n \notin \mathbb{N}$. Then either $n = 0$ or $-n \in \mathbb{N}$ by Axiom 2.1(iv). Suppose $n = 0$. Then $mn = 0 \notin \mathbb{N}$ by Axiom 2.1(iii). Suppose $-n \in \mathbb{N}$. Then $m(-n) \in \mathbb{N}$ by Axiom 2.1(ii). But $m(-n) = -(mn)$ by Proposition 1.25(iii). Since $-(mn) \in \mathbb{N}$, Proposition 2.2 implies $mn \notin \mathbb{N}$. □

Proof. (Contradiction version) Let $m \in \mathbb{N}$, $n \in \mathbb{Z}$ and suppose $mn \in \mathbb{N}$. Suppose to produce a contradiction that $n \notin \mathbb{N}$. Then either $n = 0$ or $-n \in \mathbb{N}$ by Axiom 2.1(iv). If $n = 0$ then $mn = 0$. Since $mn \in \mathbb{N}$, $0 \in \mathbb{N}$, which contradicts Axiom 2.1(iii). If $-n \in \mathbb{N}$ then

$$-(mn) = m(-n) \in \mathbb{N}$$

by Proposition 1.25(iii) and Axiom 2.1(ii). So $-(mn) \in \mathbb{N}$ and $mn \in \mathbb{N}$, contradicting Proposition 2.2. □

Proposition 2.20: For all $k \in \mathbb{N}$, $k \geq 1$.

Proof. We proceed by induction on k.

Observe that $1 \geq 1$, which establishes the base case.

Suppose for some integer $k \in \mathbb{N}$ that $k \geq 1$. Then

$$(k + 1) - 1 = k \in \mathbb{N}.$$

Hence $k + 1 > 1$ and consequently $k + 1 \geq 1$. □