Proposition 1.22:

(i) For all $m \in \mathbb{Z}$, $-(-m) = m$.

Proof. Let $m \in \mathbb{Z}$. Notice that

$$(-m) + m = 0$$

by Proposition 1.8. Also,

$$(-m) + (-(-m)) = 0$$

by the definition of additive inverses. Hence

$$(-m) + m = (-m) + (-(-m)).$$

Proposition 1.9 then implies that

$$m = -(-m).$$

□

Proposition 1.25(iii): For all $m, n \in \mathbb{Z}$,

$$(-m) \cdot n = m \cdot (-n) = -(m \cdot n).$$

Proof. Let $m, n \in \mathbb{Z}$. Then

$$\begin{aligned} (-m) \cdot n &= ((-1)m) \cdot n && \text{by Proposition 1.25(ii)} \\ &= (-1)(m \cdot n) && \text{by commutativity} \\ &= -(m \cdot n) && \text{by Proposition 1.25(ii).} \end{aligned}$$

Similarly,

$$(-n) \cdot m = -(n \cdot m).$$

Applying multiplicative commutativity to both sids of this equation we see

$$m \cdot (-n) = -(m \cdot n).$$

Hence $(-m) \cdot n = -(m \cdot n) = m \cdot (-n)$. □

Proposition 2.3: $1 \in \mathbb{N}$.

Proof. Suppose to the contrary that $1 \notin \mathbb{N}$. Then, by Axiom 2.1(iv), either $1 = 0$ or $-1 \in \mathbb{N}$. Since Axiom 1.3 tells us $1 \neq 0$, it must be that $-1 \in \mathbb{N}$. But then, from Axiom 2.1(ii), we know

$$(-1) \cdot (-1) \in \mathbb{N}.$$

Since $(-1) \cdot (-1) = 1 \cdot 1 = 1$ (Proposition 1.20), we conclude that $1 \in \mathbb{N}$. Since $1 \notin \mathbb{N}$ we have a contradiction. □

Proposition 2.5: For each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m > n$.

Proof. Let $n \in \mathbb{N}$. Let $m = n + 1$. Since $n \in \mathbb{N}$ and $1 \in \mathbb{N}$, Axiom 2.1(i) implies $m \in \mathbb{N}$. Also,

$$m - n = (n + 1) - n = 1.$$

Since $1 \in \mathbb{N}$, $m > n$. □

Proposition HW 2.1: Let m, n, and $p \in \mathbb{Z}$. If $m < n$ and $p > 0$ then

$$mp < np.$$

Proof. Let m, n, and $p \in \mathbb{Z}$ such that $m < n$ and $p > 0$. Then $n - m \in \mathbb{N}$ and $p = p - 0 \in \mathbb{N}$. Hence, by Axiom 2.1(ii), $(n - m)p \in \mathbb{N}$. But

$$(n - m)(p) = np - mp.$$

Hence $np - mp \in \mathbb{N}$ and therefore $np > mp$. □

Proposition 2.9: Let $m \in \mathbb{Z}$. If $m \neq 0$ then $m^2 \in \mathbb{N}$.

Proof. Suppose $m \in \mathbb{Z}$ and $m \neq 0$. Axiom 2.1(iv) implies that either $m \in \mathbb{N}$ or $-m \in \mathbb{N}$.

Suppose $m \in \mathbb{N}$. Then Axiom 2.1(ii) implies $m^2 = m \cdot m \in \mathbb{N}$.

On the other hand, suppose $-m \in \mathbb{N}$. Then by Proposition 1.20 and Axiom 2.1(ii),

$$m^2 = m \cdot m = (-m) \cdot (-m) \in \mathbb{N}.$$

Hence, in both cases, $m^2 \in \mathbb{N}$.

□