If A and B are sets, we say they have the *same cardinality* if there exists a bijection $f : A \to B$, in which case we write $|A| = |B|$. We say a set A is *countable* if there exists a bijection $f : \mathbb{N} \to A$.

Proposition HW14.1: Cardinality defines an equivalence relation.

Proof. Let A be a set. Consider $\mathrm{id}_A : A \to A$. Since $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$, this map is its own inverse and is hence a bijection. So $|A| = |A|$.

Suppose A and B are sets such that $|A| = |B|$. Then there exists a bijection $f : A \to B$. Since f is a bijection, there is an inverse function $f^{-1} : B \to A$ such that $f^{-1} \circ f = \mathrm{id}_A$ and $f^{-1} \circ f = \mathrm{id}_B$. Note that these equations imply that f^{-1} has an inverse, namely f. So f^{-1} is a bijection from B to A. So $|B| = |A|$.

Suppose A, B, and C are sets such that $|A| = |B|$ and $|B| = |C|$. Then there exist bijections $f : A \to B$ and $g : B \to C$. But then $g \circ f$ is a bijection from A to C. So $|A| = |C|$. □

Proposition HW14.2: The set $(0, 1)$ has the same cardinality as $(-1, 1)$.

Proof. Consider $f : (0, 1) \to (-1, 1)$ given by $f(x) = 2x - 1$. We note that if $x \in (0, 1)$, $0 < x < 1$, so $0 < 2x < 2$ and $-1 < 2x - 1 < 1$. Hence f really is a map from $(0, 1)$ to $(-1, 1)$. We claim that this map is a bijection.

Indeed, consider $g : (-1, 1) \to (0, 1)$ given by $g(y) = (y + 1)/2$. To see that g maps between these sets, suppose $y \in (-1, 1)$. Then $-1 < y < 1$ and hence $0 < y + 1 < 2$. Dividing by 2 we conclude that $0 < f(y) < 1$ as desired.

To see that $g = f^{-1}$, let $x \in (0, 1)$. Then $g(f(x)) = (2x - 1) + 1)/2 = 2x/2 = x$. Moreover, $f(g(y)) = 2g(y) - 1 = 2(y + 1)/2 - 1 = y$. So $g = f^{-1}$. Since f is invertible, it is an injection. □

Lemma HW14.A: The set $(0, 1)$ has the same cardinality as the set $(1, \infty)$.

Proof. Define $f : (0, 1) \to (1, \infty)$ by $f(x) = 1/x$. We claim that f is a map between these two sets. Indeed, suppose $0 < x < 1$. Since $x > 0$ we conclude $0 < 1 < 1/x$ as well. So $f(x) \in (1, \infty)$.

To see that f is injective, suppose $f(x_1) = f(x_2)$. Then $1/x_1 = 1/x_2$ and by taking reciprocals, $x_1 = x_2$.

To see that f is surjective, suppose $y > 1$. Proposition 8.40 implies $1 > 1/y > 0$. Let $x = 1/y$, so $x \in (0, 1)$. Then $f(x) = 1/(1/y) = y$. □

Lemma HW14.B: For all $a \in \mathbb{R}$, the set (a, ∞) has the same cardinality as the set $(0, \infty)$.

Proof. Let $a \in \mathbb{R}$. Define $f : (a, \infty) \to (0, \infty)$ by $f(x) = x - a$. . We claim that f is a map between these two sets. Indeed, suppose $x > a$. Then $f(x) = x - a > 0$. So $f(x) \in (0, \infty)$.

Define $g : (0, \infty) \to (a, \infty)$ by $g(y) = y + a$. To see that g maps between these two sets, suppose $y \in (0, \infty)$. Then $y > 0$ and $f(y) = y + a > 0 + a = a$. So $f(y) \in (a, \infty)$.

Suppose $x \in (a, \infty)$. Then $g(f(x)) = (x - a) + a = x$. Similarly, suppose $y \in (0, \infty)$. Then $f(g(y)) = (y + a) - a = y$. Hence $g = f^{-1}$ and f is a bijection. □

Proposition HW14.3: The set $(0, 1)$ has the same cardinality as $(0, \infty)$.

Proof. By Lemma 14.A, $|(0, 1)| = |(1, \infty)|$. By Lemma 14.B, $|(1, \infty)| = |(0, \infty)|$. So by transitivity of cardinality,

$$|(0, 1)| = |(0, \infty)|.$$

□

Proposition HW14.4: The set $(-1, 1)$ has the same cardinality as $\mathbb{R}$.

Proof. Let $f : (0, 1) \to (0, \infty)$ be a bijection. This map exists by problem 14.3. Define $h : (-1, 1) \to \mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & x > 0 \\ 0 & x = 0 \\ -f(-x) & x < 0. \end{cases}$$

Note that $h(x)$ is positive, negative, or zero depending on whether x is positive, negative, or zero.

To see that h is an injection, suppose $a_1, a_2 \in (0, 1)$ and $h(a_1) = h(a_2)$. Let b be the common value. Suppose $b > 0$. then a_1 and a_2 are both positive as well. Hence $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. Suppose $b = 0$. Then $a_1 = a_2 = 0$ as well. Finally, suppose $b < 0$. Then a_1 and a_2 are both positive as well. Hence $h(a_1) = -f(-a_1)$ and $h(a_2) = -f(-a_2)$. So $-f(-a_1) = -f(-a_2)$. So $f(-a_1) = f(-a_2)$. Since f is an injection, $-a_1 = -a_2$ and $a_1 = a_2$. □

Lemma HW14C: The set $(0, 1]$ has the same cardinality as the set $[1, \infty)$.

Proof. Define $f : (0, 1] \to [1, \infty)$ by $f(x) = 1/x$. We claim that f is a map between these two sets. Indeed, suppose $0 < x \leq 1$. Since $x > 0$ we conclude $0 < 1 \leq 1/x$ as well. So $f(x) \in [1, \infty)$.

To see that f is injective, suppose $f(x_1) = f(x_2)$. Then $1/x_1 = 1/x_2$ and by taking reciprocals, $x_1 = x_2$.

To see that f is surjective, suppose $y \geq 1$. If $y > 1$, Proposition 8.40 implies $1 > 1/y > 0$. And if $y = 1$, certainly $1 \geq 1/y > 0$. So $1/y \in (0, 1]$ Let $x = 1/y$, so $x \in (0, 1)$. Then $f(x) = 1/(1/y) = y$. □

Proposition HW14.5: The set $(0, 1)$ has the same cardinality as $(0, 1]$.

Proof. We will show that $(1, \infty)$ has the same cardinality as $[1, infty)$. Then, by transitivity of cardinality, using Proposition 14.3 and Lemma HW14.C,

$$|(0,1)| = |(1,\infty)| = |[1,\infty)| = |(0,1]|.$$

Define $f : [1, \infty) \to (1, \infty)$ by

$$f(x) = \begin{cases} x+1 & x \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

To see that f maps between the two sets, suppose $x \in [1, \infty)$. Then either $x = 1$ or $x > 1$. Note that $f(x)$ is either x or $x + 1$. So $f(x) \geq x$. Hence if $x > 1$, $f(x) \geq x > 1$. And if $x = 1$, then $f(x) = f(1) = 2 > 1$. So $f(x) > 1$ in all cases.

Observe that if $x \geq 1$, then $f(x) \in \mathbb{N}$ if and only if $x \in \mathbb{N}$. Indeed, suppose $x \in \mathbb{N}$. Then $f(x) = x + 1 \in \mathbb{N}$. Suppose $x \notin \mathbb{N}$. Then $f(x) = x \notin \mathbb{N}$ as well.

We now show that f is a bijection. Suppose $x_1, x_2 \in [1, \infty)$ and $f(x_1) = f(x_1)$. Either the common value b is a natural number or it is not. If $b \in \mathbb{N}$ then $x_1, x_2 \in \mathbb{N}$ and $f(x_1) = x_1 + 1$ and $f(x_2) = x_2 + 1$. So $x_1 + 1 = x_2 + 1$ and $x_1 = x_2$. Suppose $b \notin \mathbb{N}$. The $x_1, x_2 \notin \mathbb{N}$ as well and $f(x_1) = x_1$ and $f(x_2) = x_2$. So $x_1 = x_2$ in this case as well.

Now suppose $y \in (1, \infty)$. Then either $y \in \mathbb{N}$ or $y \notin \mathbb{N}$. Suppose $y \in \mathbb{N}$. Since $y > 1$, we know that $y \geq 2$. Let $x = y - 1$, so $x \geq 1$. Since $x \in \mathbb{N}$, $f(x) = x + 1 = (y - 1) + 1 = y$. Now suppose $y \notin \mathbb{N}$. Let $x = y$. Then $x \geq 1$ as $y > 1$, and $f(x) = x = y$. So f is surjective. □