Proposition 9.18: For all $m, n \in \mathbb{Z}$,

$$e(m \cdot n) = e(m) \cdot e(n).$$

Proof. Let $m \in \mathbb{Z}$. We first show that for all $n \in \mathbb{Z}_{\geq 0}$ that $e(m \cdot n) = e(m) \cdot e(n)$ by induction on n. First, when $n = 0$,

$$e(m \cdot n) = e(m \cdot 0) = e(0) = 0 = e(m) \cdot 0 = e(m) \cdot e(0) = e(m) \cdot e(n).$$

Now suppose for some $n \geq 0$ that $e(mn) = e(m)e(n)$. Then applying Proposition 9.17 and the induction hypothesis,

$$\begin{aligned} e(m(n+1)) &= e(mn + m) \\ &= e(mn) + e(m) \\ &= e(m)e(n) + e(m) \\ &= e(m)[e(n) + 1] \\ &= e(m)e(n+1). \end{aligned}$$

This completes the proof by induction.

It remains to show that for all $n < 0$, $e(mn) = e(m)e(n)$. Suppose $n < 0$. Then $-n > 0$. So

$$e(mn) = e(-(m(-n))) = -e(m(-n)) = -e(m)e(-n) = e(m)e(n)$$

where we applied Lemma 9.Â twice. □

Lemma 11.A: Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$. Let $g = \gcd(a, b)$, so $g \in \mathbb{N}$, $g \mid a$ and $g \mid b$. Then

$$\gcd\left(\frac{a}{g}, \frac{b}{g}\right) = 1.$$

Proof. Suppose $a, b \in \mathbb{Z}$ and $b \neq 0$. Let $g = \gcd(a, b)$. Since $b \neq 0$, $g > 0$. Since $g \mid a$ and $g \mid b$, and since $g \neq 0$, there are unique integers i and j such that $a = gi$ and $b = gj$; recall that we defined $a/g = i$ and $b/g = j$. Now

$$g = \gcd(a, b) = \gcd(gi, gj) = |g| \gcd(i, j) = g \gcd(i, j).$$

by Proposition 6.30, and using the fact that $g > 0$. Since $g \neq 0$, multiplicative cancellation implies $\gcd(i, j) = 1$. □

Proposition 8.50: If the sets A and B are bounded above and non-empty, and if $A \subseteq B$, then $\sup A \leq \sup B$.

Proof. Let $a \in A$. Then, since $A \subseteq B$, $a \in B$. So $a \leq \sup B$. This implies that $\sup B$ is an upper bound for A. But $\sup A$ is less than or equal to every upper bound for A. Hence $\sup A \leq \sup B$. □