This looks long, but most of the proofs are very short!

Proposition 8.A: The number $0 \in \mathbb{R}$ does not have a multiplicative inverse.

Proof. Suppose to the contrary that there exists $x \in \mathbb{R}$ such that $x \cdot 0 = 1$. Proposition 8.15 implies that $x \cdot 1 = 0$. Hence $1 = 0$, which contradicts Axiom 8.3. □

Proposition 8.B: If $c, x \in \mathbb{R}$ and $cx = 1$, then $x \neq 0$ and $c = x^{-1}$.

Proof. Suppose $c, x \in \mathbb{R}$ and $cx = 1$. Since $1 \neq 0$, by the contrapositive of Proposition 8.23, $c \neq 0$ and $x \neq 0$. Since $x \neq 0$, it has a multiplicative inverse x^{-1}. Since $cx = 1$ and $x^{-1}x = 1$,

$$cx = x^{-1}x.$$

Multiplying this equation on the right by x^{-1} and applying Axiom 8.5 to both sides we conclude that $c = x^{-1}$. □

Proposition 8.C: If $x, y \in \mathbb{R}$ and $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.

Proof. Suppose $x, y \in \mathbb{R}$ and $x \neq 0$ and $y \neq 0$. Notice that

$$(x^{-1}y^{-1})xy = (x^{-1}x)(y^{-1}y) = 1 \cdot 1 = 1.$$

Proposition 8.B then implies that $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$. □

Proposition 8.D: If $x \in \mathbb{R}$ and $x \neq 0$, then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.

Proof. Suppose $x \neq 0$. Note that

$$xx^{-1} = x^{-1}x = 1.$$

Proposition 8.B then implies that $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$. □

Proposition 8.E: If $x \in \mathbb{R}$ and $x > 0$, then $x^{-1} > 0$.

Proof. Suppose $x > 0$. We know that $x^{-1}x = 1$. Since $x > 0$ and $1 > 0$, Proposition 8.36 implies $x^{-1} > 0$ as well. □

Corollary 8.F: If $x \in \mathbb{R}$ and $x \neq 0$, if $x^{-1} > 0$ then $x > 0$.

Proof. Suppose $x \neq 0$ and $x^{-1} > 0$. Propositions 8.D and 8.E then imply that $x = (x^{-1})^{-1} > 0$. □

Proposition 8.40:

(ii) Let $x, y \in \mathbb{R}$ such that $0 < x < y$. Then $0 < 1/y < 1/x$.

Proof. Suppose $0 < x < y$. Then x^{-1} and y^{-1} are both positive as well, by Proposition 8.E. Since $x^{-1} > 0$,

$$0x^{-1} < xx^{-1} < yx^{-1}.$$

That is,

$$0 < 1 < yx^{-1}.$$

Similarly, since $y^{-1} > 0$,

$$0y^{-1} < yx^{-1}y^{-1} < yx^{-1}y^{-1}.$$

So

$$0 < x^{-1} < y^{-1}.$$

□

Proposition 8.43: Let $x, y \in \mathbb{R}$ such that $x < y$. Then there exists $z \in \mathbb{R}$ such that $x < z < y$.

Proof. Suppose $x < y$. Let $z = (x + y)/2$. Since $x < y$, $2x = x + x < x + y$. Since $2 > 0$, $2^{-1} > 0$ as well and

$$2x2^{-1} < (x + y) \cdot 2^{-1}.$$

Hence

$$x < (x + y)/2 = z.$$

Similarly, since $x < y$, $x + y < 2y$ and $z = (x + y)/2 < y$. Thus $x < z < y$. □

Proposition 8.45: If x_1 and x_2 are least upper bounds for $A \subseteq \mathbb{R}$, then $x_1 = x_2$.

Proof. Suppose x_1 and x_2 are least upper bounds for A. Then they are both upper bounds. Since x_1 is a supremum of A and since x_2 is an upper bound for A, $x_1 \leq x_2$. Since x_2 is a supremum of A and since x_1 is an upper bound for A, $x_2 \leq x_1$. So $x_1 \leq x_2 \leq x_1$ and $x_2 = x_1$. □

Proposition 8.45: $\sup((-\infty, 0)) = 0$

Proof. We need to show that 0 is an upper bound for $(-\infty, 0)$ and that for all upper bounds y, $0 \leq y$.

Let $x \in (-\infty, 0)$. Then $x < 0$. So 0 is indeed an upper bound.

Now suppose $w < 0$. By the previous proposition, there exists $z \in (0, \infty)$ such that $w < z < 0$. Hence $z \in (-\infty, 0)$ and therefore w is not an upper bound for $(-\infty, 0)$. Thus, if y is an upper bound for $(-\infty, 0)$, $y \geq 0$. □

Lemma 9.A: Suppose $f : A \to B$ and $g : B \to C$.

1. If $g \circ f$ is injective, then f is injective.
2. If $g \circ f$ is surjective, then g is surjective.
3. If $g \circ f$ is bijective, then f is injective and g is surjective.

Proof. 1) Suppose $g \circ f$ is injective. Suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. That is, $(g \circ f)(a_1) = (g \circ f)(a_2)$. Since $g \circ f$ is injective, $a_1 = a_2$.

2) Suppose $g \circ f$ is surjective. Suppose $c \in C$. Since $g \circ f$ is surjective, there exists $a \in A$ such that $(g \circ f)(a) = c$. Let $b = f(a)$. Then $g(b) = g(f(a)) = (g \circ f)(a) = c$.

3) Suppose $g \circ f$ is bijective. Then it is both injective and surjective. So by parts 1) and 2), f is injective and g is surjective. □

Proposition 9.7: (ii) If $f : A \to B$ is surjective, and $G : B \to C$ is surjective, then $g \circ f : A \to C$ is surjective.

Proof. Suppose $f : A \to B$ and $g : B \to C$ are surjective. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Since f is surjective there exists $a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = c$. □

Proposition 9.11: If $f : A \to B$ has an inverse function, the inverse function is unique.

Proof. Suppose g_1 and g_2 are inverse functions for f. Let $b \in B$. Then $f(g_1(b)) = b = f(g_2(b))$. Since f has an inverse function it is bijective and in particular injective. So $g_1(b) = g_2(b)$. Hence $g_1 = g_2$. □

Proposition 11.3: If $x, y, z \in \mathbb{R}$ with $y \neq 0$ and $z \neq 0$, then

$$\frac{xz}{yz} = \frac{x}{y}.$$

Proof. Suppose $x, y, z \in \mathbb{R}$ with $y \neq 0$ and $z \neq 0$. Then

$$\frac{xz}{yz} = (xz)(yz)^{-1} = (xz)y^{-1}z^{-1} = xy^{-1} = \frac{x}{y}.$$

□