Recall that if $\{x_n\}_{n=1}^\infty$ is a sequence of 0's and 1's, we define $s_n = \sum_{k=1}^n x_k/2^k$. We set $S = \{s_n : n \in \mathbb{N}\}$ and we define $\langle x_n \rangle_{n=1}^\infty = \sup S$. In class we showed that S is non-empty and bounded above by 1 and hence S really does have a supremum. We say that $\{x_n\}_{n=1}^\infty$ is a binary expansion of the real number $\langle x_n \rangle_{n=1}^\infty$.

In these notes we show that every number in $[0,1]$ admits a binary expansion, and that these expansions are essentially unique.

Proposition Binary 1: For every $z \in [0,1]$, there is a sequence $\{x_n\}_{n=1}^\infty$ of 0's and 1's such that

$$\langle x_n \rangle_{n=1}^\infty = z.$$

Proof. Let $z \in [0,1]$. We construct a recursively defined sequence as follows. If $z < 1/2$, we set $x_1 = 0$, otherwise $x_1 = 1$. Supposing x_1 through x_n have been defined, we set $s_n = \sum_{k=1}^n x_k/2^k$ and set

$$x_{n+1} = \begin{cases} 0 & x < s_n + \frac{1}{2^{n+1}} \\ 1 & \text{otherwise.} \end{cases}$$

We claim that for every n,

$$s_n \le x \le s_n + \frac{1}{2^n}$$

The proof is by induction on n. Suppose $n = 1$. We consider the cases $0 \le x < 1/2$ and $1/2 \le x \le 1$ separately. Suppose $0 \le x < 1/2$. Then $s_1 = 0$ and hence

$$s_1 = 0 \le x < 1/2 = 0 + 1/2 = s_1 + 1/2$$

as desired. Suppose $1/2 \le x \le 1$. Then $s_1 = 1/2$. So

$$s_1 = \frac{1}{2} \le x \le 1 = \frac{1}{2} + \frac{1}{2} = s_1 + \frac{1}{2}$$

as well. This establishes the base case.

Suppose for some $n \in \mathbb{N}$ that

$$s_n \le x \le s_n + \frac{1}{2^n}.$$

Recall that

$$x_{n+1} = \begin{cases} 0 & x < s_n + \frac{1}{2^{n+1}} \\ 1 & \text{otherwise.} \end{cases}$$

We consider two cases. Suppose $x < s_n + 1/2^{n+1}$. Then $x_{n+1} = 0$ and $s_{n+1} = s_n$. From the induction hypothesis we then conclude that

$$s_{n+1} = s_n \le x \le s_n + \frac{1}{2^{n+1}} = s_{n+1} + \frac{1}{2^{n+1}}.$$

Now suppose $x \geq s_n + 1/2^{n+1}$. Then $s_{n+1} = s_n + 1/2^{n+1}$. But then again by the induction hypothesis

$$s_{n+1} = s_n + \frac{1}{2^{n+1}} \leq x \leq s_n + \frac{1}{2^n} = s_n + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = s_{n+1} + \frac{1}{2^{n+1}}.$$

Having established the claim, we now prove that $\langle x_n \rangle_{n=1}^{\infty} = x$. Let $S = \{s_n : n \in \mathbb{N}\}$ so $\langle x_n \rangle_{n=1}^{\infty} = \sup S$. Since $s_n \leq x$ for every n, we know that x is an upper bound for S. Now suppose $y < x$. Then $x - y > 0$ and there exists $n \in \mathbb{N}$ such that $1/2^n < x - y$. Since

$$x \leq s_n + \frac{1}{2^n} < s_n + x - y$$

we may add $y - x$ to both sides of the inequality to conclude that

$$y < s_n.$$

So y is not an upper bound for S. Thus every upper bound z for S satisfies $x \leq z$ and $x = \sup S$. □

Hence we have shown that every $z \in [0, 1]$ admits a binary expansion. The next proposition shows that most of the time, different sequences lead to different numbers in $[0, 1]$. The only way two different expansions generate the same real number is when one expansion is

$$0.x_1 \cdots x_{N-1}0111 \cdots$$

and the second is

$$0.x_1 \cdots x_{N-1}1000 \cdots .$$

Proposition Binary 2: Suppose $\{x_n\}$ and $\{y_n\}$ are distinct sequences of 0's and 1's and let N be the first index where $x_n \neq y_n$, and suppose that $x_N = 0$ and $y_N = 1$. If

$$\langle x_n \rangle_{n=1}^{\infty} = \langle y_n \rangle_{n=1}^{\infty}$$

then $x_n = 1$ for all $n > N$ and $y_n = 0$ for all $n > N$.

Proof. We start by establishing some notation. Let

$$X = \langle x_n \rangle_{n=1}^{\infty}, \qquad Y = \langle y_n \rangle_{n=1}^{\infty}.$$

Let $s_n = \sum_{k=1}^{n} x_k/2^k$ and let $t_n = \sum_{k=1}^{n} x_k/2^k$, and let $S = \{s_n : n \in \mathbb{N}$ and $T = t_n : n \in \mathbb{N}$. Then $X = \sup S$ and $Y = \sup T$.

Since $x_k = y_k$ for $k < N$, and since $x_N = 0$ and $y_N = 1$,

$$T_N = s_N + \frac{1}{2^N}.$$

Suppose for some $m > N$ that $y_m = 1$ or $x_m = 0$. Then for $n > m$

$$\begin{aligned} \sum_{k=N+1}^{n} \frac{x_k - y_k}{2^k} &\leq \left[\sum_{k=N+1}^{n} \frac{1-0}{2^k} \right] - \frac{1}{2^m} \\ &= \frac{1}{2^N} \left[\sum_{k=N}^{n-N} \frac{1}{2^k} \right] - \frac{1}{2^m} \\ &< \frac{1}{2^N} \cdot 1 - \frac{1}{2^m}. \end{aligned}$$

Hence if $n > m$,

$$t_n - s_n = t_N - s_N - \sum_{k=N+1}^{n} \frac{x_k - y_k}{2^k} = \frac{1}{2^N} - \sum_{k=N+1}^{n} \frac{x_k - y_k}{2^k} > \frac{1}{2^N} - \frac{1}{2^N} + \frac{1}{2^m} = \frac{1}{2^m}.$$

That is, if $n > m$, then

$$t_n > s_n + \frac{1}{2^m}.$$

Since Y is the supremum of the set T, $Y \geq t_n$ for all n and

$$Y \geq s_n + \frac{1}{2^m}$$

for all $n > m$. Since the sequence $\{s_n\}$ is increasing we conclude that this same inequality holds for all $n \in \mathbb{N}$ hance hence $Y - \frac{1}{2^m}$ is an upper bound for the set S. But then

$$Y - \frac{1}{2^m} \geq \sup S = X$$

and therefore $Y > X$. Thus, if $Y = X$, we conclude that for all $n > N$, $x_n = 1$ and $y_n = 0$. □