1. Suppose $f : X \to X$ is Lipshitz continuous with Lipschitz constant $K < 1$. Show that if Y is complete, then there is a unique $x \in X$ solving $f(x) = x$. Hint: Starting from any x_0, let $x_{n+1} = f(x_n)$. Show the sequence is Cauchy.

2. Show that a space X is topologically compact if and only if whenever $\{F_\alpha\}_{\alpha\in I}$ is a family of closed sets in X with the finite intersection property, $\cap F_\alpha \neq \varnothing$.

3. In this problem, we will seek a solution to the initial value problem

$$\begin{aligned} f'(t) &= F(t, f(t)) \\ f(0) &= a \end{aligned}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}$.

To obtain the existence result, we need to assume that F is sufficiently nice; we will assume that F is continuous, and moreover that there exists a constant K such that

$$|F(x, y_1) - F(x, y_2)| \leq K|y_1 - y_2|$$

for all $x, y_1, y_2 \in \mathbb{R}$.

Define $G : C[-T, T] \to C[-T, T]$ by

$$G(f)(t) = a + \int_0^t F(s, f(s))\, ds.$$

a) Explain why $G(f) \in C[-T, T]$ if $f \in C[-T, T]$.

b) Show that if f solves the initial value problem for $t \in [-T, T]$, then $G(f) = f$.

c) Show that G is Lipschitz with Lipschitz constant TK.

d) Assuming $T < 1/K$, show that there exists a solution of $G(f) = f$ defined for $t \in [-T, T]$. You may use the fact that $C[-T, T]$ is complete; we'll show this later.

e) Assuming $T < 1/K$, show that there exists a unique solution of the initial value problem defined on $(-T, T)$.

f) Extra credit: Show that there exists a solution f of the initial value problem defined for all $t \in \mathbb{R}$.

4. Carothers 7.32

5. Carothers 8.13

6. Carothers 8.16

7. Carothers 8.17

8. Show that $A \subseteq \mathbb{R}^n$ is compact in the ℓ_∞ norm if and only if it is closed (with respect to the ℓ_∞ norm) and bounded (with respect to the ℓ_∞ norm).

What does this say about compactness with respect to the ℓ_2 norm?

9. Carothers 8.29

10. Carothers 8.38

11. Carothers 8.40

12. Carothers 8.78