1. D&M 3.19

2. Suppose $f \in L^1_{\text{loc}}(\mathbb{R})$ and ϕ is a continuously differentiable function with compact support. Define

$$(f * \phi)(x) = \int_{\mathbb{R}} \phi(x-y) f(y)\, dy;$$

this is the **convolution** of ϕ and f. Show that $f * \phi$ is differentiable and

$$(f * \phi)'(x) = \int_{\mathbb{R}} \phi'(x-y) f(y)\, dy.$$

Conclude that if ϕ is an infinitely differentiable function with compact support, then $f * \phi$ is infinitely differentiable. *Hint:* Use the Dominated Convergence Theorem.

3. Suppose ϕ is a continuously differentiable function on $[0,1]$ such that $\phi(0) = 0$. Show that there is a constant $c > 0$ such that

$$\int_0^1 \phi'(x)^2\, dx \geq c \int_0^1 \phi(x)^2\, dx.$$

4. We define $H^1_0((0,1))$ to be the closure of $\mathcal{D}((0,1))$ in the H^1 norm. Show that the bilinear form

$$\varphi(u,v) = \int_0^1 u'(x)v'(x)\, dx$$

is coercive on $H^1_0([0,1])$.

5. Let $f \in L^2([0,1])$. Show that there is a unique function $u \in H^1_0([0,1])$ such that

$$\int_0^1 u'\psi' = \int_0^1 f\psi$$

for all $\psi \in \mathcal{D}((0,1))$. You may assume that $H^1((0,1))$ is complete.

6. There exists an infinitely differentiable function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi \geq 0$, the support of ϕ is contained in $[-1,1]$, and $\int_{\mathbb{R}} \phi = 1$. You do not need to prove this. Instead, prove:

a) For each $n \in \mathbb{N}$, let $\phi_n(x) = n\phi(nx)$. Show that ϕ_n all the aforementioned properties that ϕ has, except that the support of ϕ_n in contained in $[-1/n, 1/n]$.

b) Suppose $g \in L^\infty(\mathbb{R})$. Let $g_n = \phi_n * g$. Show $||g_n||_\infty \leq ||g||_\infty$.

c) Suppose $g \in L^1_{\text{loc}}(\mathbb{R})$ and is continuous at some $x \in \mathbb{R}$. Show that $g_n(x) \to g(x)$. Hint: $g(x) = \int_{\mathbb{R}} \phi_n(x-y) g(x)\, dy$.