1. (Solution by David Maxwell)
Let $\tilde{X}$ be a completion of the normed space X, and let $\phi : X \to \tilde{X}$ be an isometry such that $\phi(X)$ is dense in $\tilde{X}$. Given $\tilde{x}, \tilde{y} \in \tilde{X}$, let (x_n) and (y_n) be sequences in X such that $\phi(x_n) \to \tilde{x}$ and $\phi(y_n) \to \tilde{y}$. We define

$$\tilde{x} + \tilde{y} = \lim_{n\to\infty} \phi(x_n + y_n)$$

and

$$\lambda\tilde{x} = \lim_{n\to\infty} \phi(\lambda x_n).$$

a) Show that these limits exist and are independent of the choice of approximating sequences.

b) Convince yourself that it is then easy and tedious to verify $\tilde{X}$ with these operations is indeed a vector space (if you decide prove this, don't hand it in!).

c) Show that the distance function on the metric space $\tilde{X}$ is indeed a norm.

d) Show that ϕ is a continuous linear map.

2. (Solution by Lyman Gilispie)
Show that the completion of an inner product space X is a Hilbert space. That is, if $\phi : X \to \tilde{X}$ is an isometry with dense image into the complete space $\tilde{X}$, then $\tilde{X}$ with the normed space structure described in problem 1 is in fact an inner product space, and that if $x, y \in X$,

$$\langle x, y\rangle_X = \langle \phi(x), \phi(y)\rangle_{\tilde{X}}\,.$$

Don't do a lot of work.

3.

a) Let $\phi(x) = \chi_{[0,1)}$. Prove or disprove: $\phi \in H^1((-1,1))$.

b) For which values of $\alpha \in (0,1]$ is

$$|x|^\alpha \in H^1((-1,1))?$$