1. (Solution by David Maxwell)
Suppose X is a normed space and $W \subseteq X$ is a subspace. Show that $\overline{W}$ is a subspace.

2. (Solution by TJ Barry)
Suppose X is a Banach space. Let $I : X \to X$ be the identity operator.

a) Show that if $T \in \mathcal{B}(X, X)$ and $||I - T|| < 1$, then T is invertible. Hint: Consider the Taylor series for $1/(1 - x)$.

b) Show that the set of invertible elements of $\mathcal{B}(X, X)$ is open.

3. (Solution by Lyman Gilispie)
Let X and Y be vector spaces and let $\mathcal{B} = \{x_\alpha\}_{\alpha \in I}$ be a basis for X. Given a map $F : \mathcal{B} \to Y$, show that there exists a unique linear map $T : X \to Y$ such that $T|_{\mathcal{B}} = F$.

Moral: A linear map is completely specified by it is action on a basis.

4. (Solution by David Maxwell)

a) Given an n-dimensional vector space X over $\mathbb{R}$, show that there exists a (real) linear isomorphism between X and $\mathbb{R}^n$.

b) If X is a real or complex normed vector space and $T : \mathbb{R}^n \to X$ is an injective real-linear map, show that $|| \cdot ||_T$ on $\mathbb{R}^n$ defined by $||x||_T = ||Tx||_X$ is a norm on $\mathbb{R}^n$.

c) Show that if X is a real or complex normed vector space, then any two norms on X are equivalent.

5. (Solution by Vikenty Mikheev)
Suppose X is a finite dimensional normed vector space and W is a proper subspace of X. Show that there exists an $x \in X$ with $||x|| = 1$ and $d(x, w) \ge 1$ for all $w \in W$.

6. (Solution by Will Mitchell)
Suppose $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Given $y \in \ell_q$, and $x \in \ell_p$ let

$$T_y(x) = \sum_{k=1}^{\infty} x_k \overline{y_k}.$$

We will prove in class that T_y is well defined, linear, and continuous.

a) Show that $||T_y|| = ||y||_{\ell_q}$ for all $y \in \ell_q$.

b) Given $S \in \mathcal{B}(\ell_p, \mathbb{C})$, show that $S = T_y$ for some $y \in \ell_q$.

c) Show that ℓ_p is complete for all $p \in (1, \infty)$.