1. Suppose α, β, and γ are continuous on $[a,b]$ and $\alpha \neq 0$. Given $x_0, v_0 \in \mathbb{R}$, and $f \in C[a,b]$, show that there exists a unique $u \in C^2[a,b]$ such that

$$\alpha u'' + \beta u' + \gamma u = f$$

satisfying $u(b) = x_0$ and $u'(b) = v_0$.

2. Suppose $\{u_k\}$ is an orthonormal basis for a Hilbert space X, let $\{b_k\} \in \ell_2$, and let $\{c_k\}$ be a bounded sequence in $C^2[a,b]$. Show that

$$v(t) = \sum_{k=1}^{\infty} b_k c_k(t) u_k$$

is a well defined function from $\mathbb{R}$ to X Moreover, it is differentiable and

$$v'(t) = \sum_{k=1}^{\infty} b_k c_k'(t) u_k$$

You should decide for yourself what differentiability means in this context.

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4. Let $T : L^2_\omega \to L^2_\omega$ be the compact operator we constructed in class for solving the equation

$$-\frac{1}{\omega} Lu = f \tag{1}$$

together with separated boundary conditions. We showed that if f is continuous then Tf is a C^2 solution of this problem. Now show that if $f \in L^2_\omega$, then u solves equation (1) in the sense of distributions, and that $u \in H^2([a,b])$. Hint: approximate f with continuous functions.

5. Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $f \in L^1_{\text{loc}}(\Omega)$. Show that the map $T_f : \mathcal{D}(\Omega) \to \mathbb{F}$ given by

$$T_f(\phi) = \int_\Omega f\phi$$

is an element of $\mathcal{D}'(\Omega)$.

The following problem is part of the take-home final. You can start thinking about it now.

6. Suppose $p \in C^1[a,b]$. Show that given $f \in L^2[a,b]$, and $T > 0$, there exists a function $u : [0,T] \to L^2[a,b]$ such that

1. $u(0) = f$.
2. u is continuous on $[0,T]$.
3. u is differentiable on $(0,T]$.
4. $\frac{d}{dt}u = (pu')'$; the derivative on the right is in the sense of distributions.
5. For all $t > 0$, $u(t,a) = 0 = u(t,b)$.