Let $a, b \in \mathbb{R}$ with $a < b$. Let $A \in C[a,b] \otimes \mathbb{F}^{n\times n}$, let $f \in C[a,b] \otimes \mathbb{F}^n$, and let $x_0 \in \mathbb{F}^n$. We would like to show that there exist a unique solution in $C^1[a,b] \otimes \mathbb{F}^n$ of

$$x' = Ax + f$$

with the initial condition $x(a) = x_0$. This homework guides you through a (likely) unfamiliar approach to showing this is true.

The usual approach to showing solutions of differential equations exist is done using a contraction mapping argument, but this only ensures existence of solutions on a small time interval. There is then an argument to show that if the right-hand side is nice enough, you can extend the solutions to a large time interval. For linear equations, like the one studied here, one can always extend. The method of proof demonstrated here exploits linearity to obtain an existence on the whole time interval without a continuation argument.

1. Suppose $u \in C^1[a,b]$, $\alpha \in C^0[a,b]$, and suppose

$$u' \leq \alpha u$$

on $[a,b]$. Note that u is **real** valued, not $\mathbb{F}^n$-valued.

Let $v(x) = \exp(\int_a^x \alpha(s)\, ds)$. Show that $v' = \alpha v$, $v(a) = 1$ and that

$$u \leq u(a)v$$

on $[a,b]$. Hint: Observe that $v \neq 0$ and consider u/v.

Moral: The solution of $w' = \alpha w$ with $w(a) = u(a)$ dominates u.

2. Suppose $x \in C^1([a,b]) \otimes \mathbb{F}^n$ and

$$x' = Ax$$

on $[a,b]$ and $x(a) = 0$. Show that $x \equiv 0$. Hint: Let $z = ||x||^2$ and show that z satisfies a differential inequality of the form $z' \leq \alpha z$ on $[a,b]$. Then apply Problem 1.

3. Given $v \in L^2[a,b] \otimes \mathbb{F}^n$ let

$$Tv(x) = \int_a^b K(x,y)v(y)\, dy$$

where $K(x,y) = A(y)$ if $y < x$ and $K(x,y) = 0$ otherwise. Show that

a) If $v \in L^2[a,b] \otimes \mathbb{F}^n$, $Tv \in C[a,b] \otimes \mathbb{F}^n$.

b) T is a compact map from $L^2[a,b] \otimes \mathbb{F}^n$ to itself.

c) If $v \in C[a,b] \otimes \mathbb{F}^n$ and $u = Tv$, then $u \in C^1[a,b] \otimes \mathbb{F}^n$ and $u' = Av$.

4. Show that if $u \in L^2[a,b] \otimes \mathbb{F}^n$ and $Tu = u$ then $u = 0$. Hint: Be sure to first show that $u \in C^1[a,b] \otimes F^n$.

5. Problem 2 from the previous assignment holds even when K is not self adjoint; you'll show this in future work. Assuming this for the moment, show that given $g \in L^2[a,b] \otimes \mathbb{F}^n$ there exists a unique solution $u \in L^2[a,b] \otimes \mathbb{F}^n$ of

$$u - Tu = g.$$

6. Finish the job.