A recursively defined sequence $(x_k)_{k=m}^{\infty}$ is defined by giving an initial value $x_m = a$ and then giving a formula for how to compute x_{n+1} given that x_n has been computed. For example, if $m = 0$, $a = 1$, and if $x_{n+1} = x_n \cdot (n + 1)$ once x_n has been computed, then

$$\begin{aligned} x_0 &= 1 \\ x_1 &= x_0 \cdot (0 + 1) = 1 \cdot 1 = 1 \\ x_2 &= x_1 \cdot (1 + 1) = 1 \cdot 2 = 2 \\ x_3 &= x_2 \cdot (2 + 1) = 1 \cdot 2 \cdot 3 = 6 \\ x_4 &= x_3 \cdot (3 + 1) = 1 \cdot 2 \cdot 3 \cdot 4 = 24. \end{aligned}$$

You are probably familiar with the definition of $k! = 1 \cdot 2 \cdots (k - 1) \cdot k$. The recursively defined sequence above gives a way to define $k!$ without resorting to the mathematically vague '$\cdots$'.

The purpose of this document is to give a rigorous justification of the construction of recursively defined sequences. We have the following standing assumptions:

- X is a set of possible sequence values (e.g. $X = \mathbb{Z}$ for an integer-valued sequence),
- $m \in \mathbb{Z}$, which will be the initial index of the sequence,
- $a \in X$, which will be the initial value of the sequence, and
- $f : \mathbb{Z}_{\geq m} \times X \to X$ is the defining function for the sequence.

We want to build an X-valued sequence $(x_k)_{k=m}^{\infty}$ such that

(i) $x_m = a$

(ii) $x_{n+1} = f(n, x_n)$ for all $n \in \mathbb{Z}_{\geq m}$.

Notice that the function f plays the role of the 'formula' for defining the next term of the sequence given that the previous term has been computed. In the factorial example, $f(n, y) = y \cdot (n + 1)$.

The bulk of the work for building the infinite sequence is to build some finite recursively defined sequences. For the factorial example one could build by hand the following finite sequences: (1), $(1, 1)$, $(1, 1, 2)$, $(1, 1, 2, 6)$, $(1, 1, 2, 6, 24)$. Each of these finite sequences satisfies the initial condition (i) and the recursion condition (ii), except that the recursion condition holds only for a finite number of indices. We will call such a sequence an initial matching sequence. More formally, if $n \in \mathbb{Z}_{\geq m}$, and $(y_k)_{k=m}^{n}$ is a finite sequence, we say that it is an **initial matching sequence to index** n if

(i) $y_m = a$

(ii) $y_{k+1} = f(k, y_k)$ for all $k \in \mathbb{Z}$ such that $m \leq k \leq n - 1$.

Note that for an initial matching sequence, the recursion condition (ii) holds for all indices beyond the first one, but there are only finitely many such indices.

Using mathematical induction we now show that we can always build an initial matching sequence of whatever length we please.

Proposition R1: For each $n \in \mathbb{Z}_{\geq m}$ there exists an initial matching sequence to index n.

Proof. We proceed by induction on $n \geq m$. Note that the sequence with one value $y_m = a$ is an initial matching sequence to index m. This establishes the base case.

Suppose for some $n \in \mathbb{Z}_{\geq m}$ that $(y_k)_{k=m}^n$ is an initial matching sequence to index n. Define

$$z_k = \begin{cases} y_k & m \leq k \leq n \\ f(n, y_n) & k = n+1. \end{cases}$$

We claim that $(z_k)_{k=m}^{n+1}$ is a matching sequence to index $n+1$. To do this we must show that

(i) $z_m = a$

(ii) $z_{k+1} = f(k, z_k)$ for all $k \in \mathbb{Z}$ such that $m \leq k \leq (n+1)-1$ (i.e. $m \leq k \leq n$).

Note that $z_m = y_m = a$, so condition (i) is satisfied. Suppose that $k \in \mathbb{Z}$ and $m \leq k \leq n$. Then either $m \leq k \leq n-1$ or $k = n$. If $m \leq k \leq n-1$ then $m \leq k < k+1 \leq n$ and hence $z_{k+1} = y_{k+1}$ and $z_k = y_k$. Moreover, since $(y_k)_{k=1}^n$ is an initial matching sequence,

$$z_{k+1} = y_{k+1} = f(k, y_k) = f(k, z_k).$$

Hence the recursion condition (ii) hold for $m \leq k \leq n-1$. However,

$$z_{n+1} = f(n, y_n) = f(n, z_n)$$

by the definition of $(z_k)_{k=m}^{n+1}$. So the recursion condition (ii) also holds when $k = n$. Thus $(z_k)_{k=m}^n$ is an initial matching sequence to index $n+1$. □

At this stage we know that we can always build a matching sequence up to any index we want. It is intuitively clear that there is only one way to build an initial matching sequence to a given length. The key tool to show this rigorously is the Well Ordering Principle (which was itself a consequence of the Induction Axiom). Here's the proof.

Proposition R2: Suppose for some $n \in \mathbb{Z}_{\geq m}$ that $(y_k)_{k=m}^n$ and $(\hat{y}_k)_{k=m}^n$ are matching sequences to index n. Then $y_k = \hat{y}_k$ for all $m \leq k \leq n$.

Proof. Let $A = \{k \in \mathbb{Z} : m \leq k \leq n \text{ and } y_k \neq \hat{y}_k\}$. The proposition is proved if we can show that A is empty. Suppose to the contrary that A is not empty. Since it is bounded below by m, Proposition 2.33 implies that A has a least element r. Note that $r \neq m$ since $y_m = a = \hat{y}_m$.

Hence $m < r \leq n$. Let $s = r - 1$, so $m \leq s \leq n - 1$. Since both sequences are initial matching sequences,

$$y_{s+1} = f(s, y_s) \quad \text{and}$$
$$\hat{y}_{s+1} = f(s, \hat{y}_s).$$

Since $s < r$, $s \notin A$ and hence $y_s = \hat{y}_s$. So

$$y_{s+1} = f(s, y_s) = f(s, \hat{y}_s) = \hat{y}_{s+1}.$$

That is, $y_r = \hat{y}_r$. But $r \in A$, so $y_r \neq \hat{y}_r$. This is a contradiction. □

Combining Propositions R1 and R2 we obtain the following

Proposition R3: For every $n \in \mathbb{Z}_{\geq m}$ there exists exactly one initial matching sequence up to index n.

We will denote the (unique) matching sequence up to index n by $\left(y_k^n\right)_{k=m}^n$. Note that the raised index is a label, not an exponent. For the factorial example,

$$(y_k^0)_{k=0}^0 = (y_0^0) = (1)$$
$$(y_k^1)_{k=0}^1 = (y_0^1, y_1^1) = (1, 1)$$
$$(y_k^2)_{k=0}^2 = (y_0^2, y_1^2, y_2^2) = (1, 1, 2)$$
$$(y_k^3)_{k=0}^3 = (y_0^3, y_1^3, y_2^3, y_3^3) = (1, 1, 2, 6)$$

and so forth. Observe that given two of these sequences, the short sequence agrees with the long sequence for all of its entries. The following proposition shows that this is always the case.

Proposition R4: If p and q are integers such that $m \leq p \leq q$ then

$$y_k^p = y_k^q$$

for all integers k such that $m \leq k \leq p$.

Proof. Consider $(y_k^q)_{k=m}^p$, which is the long sequence $(y_k^q)_{k=m}^q$ truncated to the length of the short sequence $(y_k^p)_{k=m}^p$. If we can show that $(y_k^q)_{k=m}^p$ is an initial matching sequence up to index p, then Proposition R3 will imply that $y_k^q = y_k^p$ for all $m \leq k \leq q$.

Note that $y_m^q = m$. Moreover, suppose that $m \leq k \leq p - 1$. Then $m \leq k \leq q - 1$ as well and hence, since $(y_k^q)_{k=m}^q$ is an initial matching sequence to index q,

$$y_{k+1}^q = f(k, y_k^q).$$

This proves that $(y_k^q)_{k=m}^p$ is an initial matching sequence to index p. □

We now have all the ingredients required to justify the construction of recursively defined sequences.

Theorem R5 (Principle of Recursive Definition): Let X be a set and suppose $m \in \mathbb{Z}$. Given a function $f : \mathbb{Z}_{\geq m} \times X \to X$ and an initial value $a \in X$ there exists a unique X-valued sequence $(x_k)_{k=1}^{\infty}$ such that

(i) $x_m = a$

(ii) $x_{n+1} = f(n, x_n)$ for all $n \in \mathbb{Z}_{\geq m}$.

Proof. We define a sequence $x_k = y_k^k$, so x_k is the last value of the initial matching sequence up to index k. Note that $x_m = y_m^m = a$. Now consider some $n \in \mathbb{Z}_{\geq m}$. Then

$$x_{n+1} = y_{n+1}^{n+1} = f(n, y_n^{n+1})$$

by the definition of x_{n+1} and since $(y_k^{(n+1)})_{k=m}^{n+1}$ is an initial matching sequence. But $y_n^{n+1} = y_n^n$ by Lemma R4 and $y_n^n = x_n$ by the definition of x_n. Hence

$$x_{n+1} = f(n, x_n).$$

Thus the sequence $(x_k)_{k=m}^{\infty}$ satisfies conditions (i) and (ii). This proves existence.

Now consider two such sequences $(x_k)_{k=m}^{\infty}$ and $(\hat{x}_k)_{k=m}^{\infty}$. Let $n \in \mathbb{Z}_{\geq m}$. Observe that $(x_k)_{k=m}^{n}$ and $(\hat{x}_k)_{k=m}^{n}$ are both initial matching sequences up to index n. By Proposition R2 they are the same, and in particular $x_n = \hat{x}_n$. Hence for all $n \in \mathbb{Z}_{\geq m}$, $x_n = \hat{x}_n$ and the sequences are the same. □