Problem 1.21
Show that a number $x \in [0,1]$ has more than one p-adic expansion if and only if $x = \sum_{k=1}^{n} a_k/p^k$ for some n where $a_n \neq 0$. Show also that in this case x has exactly one other expansion,

$$x = \sum_{k=1}^{n-1} a_k/p^k + \frac{a_n - 1}{p^n} + \sum_{k=n+1}^{\infty} \frac{p-1}{p^n}. \tag{1}$$

Also, characterize the numbers in $[0,1]$ with repeating and eventually repeating p-adic expansions.

Solution
Suppose x has two different expansions,

$$\begin{aligned} x &= \sum_{k=1}^{\infty} \frac{a_k}{p^k} \\ &= \sum_{k=1}^{\infty} \frac{b_k}{p^k}. \end{aligned}$$

Let N be the index in which they first differ, and without loss of generality assume that $a_N > b_N$. We will show that $a_N = b_N + 1$, $a_n = 0$ for $n > N$, and $b_n = p-1$ for $n > N$. This will prove that if x has two different expansions, then one must be a terminating expansion, and that the only other expansion is the one of the form (1).

Let $y = \sum_{k=1}^{N-1} \frac{b_k}{p^k}$. Then

$$\begin{aligned} x &\le \sum_{k=1}^{N-1} \frac{b_k}{p^k} + \frac{b_N}{p^N} + \sum_{k=N+1}^{\infty} \frac{p-1}{p^k} \\ &= y + \frac{b_N}{p^N} + \frac{p-1}{p^{N+1}} \frac{p}{p-1} \\ &= y + \frac{b_N}{p^N} + \frac{1}{p^N} \end{aligned}$$

with strict inequality unless $b_n = p-1$ for all $n > N$. Similarly,

$$\begin{aligned} x &\ge \sum_{k=1}^{N-1} \frac{a_k}{p^k} + \frac{a_N}{p^N} + \sum_{k=N+1}^{\infty} \frac{0}{p^k} \\ &= y + \frac{a_N}{p^N} \end{aligned}$$

with strict inequality unless $a_n = 0$ for $n > N$. These inequalities together imply

$$\frac{a_N}{p^N} \le \frac{b_N + 1}{p^N}$$

and hence $a_N \le b_N + 1$ (with strict inequality unless $b_n = p-1$ and $a_n = 0$ for $n > N$). But $a_N \ge b_N + 1$ since $a_N > b_N$ and since a_N and b_N are integers. Hence $a_N = b_N + 1$ and $b_n = p-1$ and $a_n = 0$ for $n > N$.

If x has a terminating p-adic expansion, then x is of the form

$$x = \frac{a}{p^N}$$

where $N \in \mathbb{N}$ and $0 \leq a \leq p^N$.

If x has a repeating p-adic expansion, then x is of the form

$$\frac{a}{p^N - 1}$$

where $N \in \mathbb{N}$ and where $0 \leq a \leq p^N - 1$.

If x has an eventually repeating p-adic expansion, then x can be decomposed into a terminating and a repeating part. So x is of the form

$$x = \frac{a}{p^N} + \frac{1}{p^N} \frac{b}{p^M - 1}$$

where $N, M \in \mathbb{N}$, $0 \leq a < p^N$, and $0 \leq b \leq p^M - 1$. This is easily seen to be the same as

$$x = \frac{d}{p^N(p^M - 1)}$$

for some d with $0 \leq d \leq p^N(p^M - 1)$. It is perhaps surprising that every rational admits an expression of this form.