Overview

In this lab you will investigate two phenomena (resonance and beats) related to periodic forcing of second-order linear differential equations.

In this first part of the lab, we look at the phenomena of beats. Consider the equation

$$x'' + \omega^2 x = cos(\beta t).$$

1. Find the general solution of the associated homogeneous equation.

2. What is the natural frequency of this system?

3. Suppose $\beta \neq \omega$. Find a particular solution x_p of the differential equation.

4. Suppose $\beta = \omega$. Find a particular solution x_p of the differential equation.

5. Using Octave, graph the solution x_p you found in the previous step assuming $\omega = 55$.

6. On a single graph, plot your particular solutions for $\beta = 60$, $\beta = 50$, $\beta = 45$, and $\beta = 40$, all assuming $\omega = 55$.

7. Assuming $\omega \neq \beta$, what initial condition does your particular solution x_p satisfy at time $t = 0$?

8. Assuming $\omega \neq \beta$, find a solution with the initial conditions $u(0) = u'(0) = 0$.

9. Use the trigonometric identity $2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B)$ to write your answer to the previous problem as a product of sines.

10. Assuming $\omega = 55$ and $\beta = 45$, make a plot of your solution to question ???.

11. You can interpret your solution as a high frequency sinusoid modulated by a low frequency sinusoidal envelope. Add to your previous graph the low frequency envelope.

In this second part of the lab, we look at the interaction of friction and resonance. Consider the following equation

$$x'' + cx' + x = \cos(\beta t)$$

12. Suppose $c < 2$. What frequency do solutions of the associated homogeneous equation oscillate at?

13. Suppose $c > 2$. What frequency do solutions of the associated homogeneous equation oscillate at?

14. Find a particular solution if $c = 0$ and $\beta = 1$.

15. Show that if $c \neq 0$, a particular solution of the non-homogeneous equation is

$$x_p(t) = \frac{1-\beta^2}{(1-\beta^2)^2 + (c\beta)^2}\cos(\beta t) + \frac{c\beta}{(1-\beta^2)^2 + (c\beta)^2}\cos(\beta t).$$

16. The previous solution can be written in the form $x_p = C\cos(\beta t)$. Show that

$$C = \frac{1}{\sqrt{(1-\beta^2)^2 + (c\beta)^2}}$$

17. We'll define

$$R(\beta) = \frac{1}{\sqrt{(1-\beta^2)^2 + (c\beta)^2}}$$

Then $R(\beta)$ is the amplitude of the particular solution x_p when the system is forced at frequency β and has friction coefficient c. Make a graph of $R(\beta)$ for each of the following values of c: 0.1, 0.5, 1, 2. Your graph domain should be $0 \leq \beta \leq 3$.

18. Practical resonance occurs if $R(\beta)$ has a maximum at a frequency $\beta \neq 0$. Mark, by hand, the places on your graphs in the previous problem where practical resonance occurs. Do a computation to show that if $c = 1$, then practical resonance occurs at $\beta = \sqrt{1/2}$.

19. Show that if $c < \sqrt{2}$, then practical resonance occurs at $\beta = \sqrt{1 - \frac{c^2}{2}}$.

20. Does the practical resonance occur at the system's natural frequency?

21. Show that if $c > \sqrt{2}$, then no practical resonance occurs.

22. For what values of c does the system exhibit oscillatory solutions, but no practical resonance?