We are considering a commercially valuable species that can be harvested. In the absence of harvesting, the species grows according to the logistic equation

$$p' = kp(1 - p/M).$$

In this lab, we compare two different harvesting schemes. In the first scheme, we harvest a fixed number of individuals (H) per unit time. The equation is

$$p' = kp(1 - p/M) - H$$

In the second scheme, we harvest a fixed proportion h of the individuals per unit time. If the current population is p, the harvest rate is hp, and the equation is

$$p' = kp(1 - p/M) - hp.$$

Making the change of units $Q = P/M$ and $\tau = kt$ these equations take on the dimensionless forms

$$\frac{dQ}{d\tau} = Q(1 - Q) - a \tag{1}$$

and

$$\frac{dQ}{d\tau} = Q(1 - Q) - bQ \tag{2}$$

where $a = H/(kM)$ and $b = (h/k)$.

For the following questions, you do **not** need to type anything up – handwritten answers are ok, but you must be tidy. Answer the questions clearly and succinctly. All plots and sketches are to be done by hand, and can only include physically relevant values of any variables or parameters. For questions that ask for the number of equilibria in various scenarios, you must justify your answer with a brief, tidy computation.

Good luck!

Consider first the case of equation (1).

1. Why should we only consider $a \geq 0$?

2. How many equilibria are there if $a > 1/4$?

3. There is one equilibrium if $a = 1/4$. What is its value (what is the value of Q for this equilibrium)? What is its stability? What values of H and P does this equilibrium correspond to?

4. How many equilibria are there if $0 \leq a < 1/4$? What are their values and stability?

5. Draw (by hand) representative time series for each of the these three cases ($0 \leq a < 1/4$, $a = 1/4$, $a > 1/4$). Your sketches should include all equilibrium solutions and all relevant representative solutions (i.e. above all equilibria, between any equilibria, and below all equilibria). Your sketch should only include physically relevant values for any variables.

6. For what values of a is it possible for a species to go extinct with this harvesting scheme?

7. Draw (by hand) a bifurcation diagram. Label any equilibria as stable or unstable.

8. What happens to the equilibria as a gets close to 1/4?

9. Why might it be a bad idea to harvest at a rate of $a < 1/4$ but very near 1/4?

Now consider the case of equation (2)

10. Why should we only consider $b \geq 0$?

11. How many equilibria are there for $b > 1$ (i.e. $h > k$)? What are their values and stabilities?

12. How many equilibria are there for $b = 1$ (i.e. $h > k$)? What are their values and stabilities?

13. How many equilibria are there for $0 \leq b < 1$ (i.e. $h > k$)? What are their values and stabilities?

14. Draw (by hand) representative time series for each of these three cases.

15. For what values of b is it possible for a species to go extinct with this harvesting scheme?

16. Draw (by hand) a bifurcation diagram. Label any equilibria as stable or unstable. Only include physically relevant values of Q and b. Label any equilibria as stable or unstable.

17. If you harvest at a per-capita rate $b < 1$, what will the long-term population size be? What will the long term harvest rate be?

18. What value of b maximizes the harvest rate (in the long term)? What is this long term rate of harvest in terms of the original variables (P, M, etc.)?

19. If you harvest at this optimal choice of b, is it possible for the species to go extinct?

20. In a brief paragraph, summarize the advantages and disadvantages of the two harvesting schemes.