1. Let $\mathbf{x}$ be a chart. Define a family of charts $\mathbf{x}_\lambda(u,v)$ by

$$\mathbf{x}_\lambda(u,v) = \mathbf{x}(u,v) + \lambda U(u,v).$$

a) Let E_λ, F_λ, and G_λ be the coefficients of the first fundamental form of $\mathbf{x}_\lambda$. Define $E' = \frac{d}{d\lambda}|_{\lambda=0} E_\lambda$, and similarly for F' and G'. Show that

$$E' = -2l \qquad F' = -2m \qquad G' = -2n.$$

b) Prove that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} (EG - F^2) = -4H(EG - F^2).$$

c) Let R be a region in the domain of $\mathbf{x}$ and let A_λ be its image under $\mathbf{x}_\lambda$. Show that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \text{Area}(A_\lambda) = \int_R -2H\sqrt{EG - F^2}\, du\, dv.$$

2. Let M be the surface of revolution obtained by revolving the curve $(f(t), g(t), 0)$ about the x-axis, where $a < t < b$. Show that the area of M is

$$2\pi \int_a^b g(t)\, dt.$$

Use this computation to compute the area of a torus with outer radius R and inner radius r.

3. Let M be a surface that is the image of a Monge patch $\mathbf{x}(u,v) = (u, v, f(u,v))$ where $(u,v) \in D$. Determine the area of M in terms of an integral over D involving f.

Use this computation to compute the area of the surface $z = x^2 + y^2$ where D is the unit disc.

4. Let α be a regular but not-necessarily unit speed curve in M. Show that the normal and geodesic curvatures of α can be computed via

$$k_n = \frac{\langle \alpha'', U\rangle}{\langle \alpha', \alpha'\rangle}$$
$$k_g = \frac{\langle \alpha'', U \times \alpha'\rangle}{[\langle \alpha', \alpha'\rangle]^{3/2}}.$$

Hint: Recall that a unit speed reparameterization β of α satisfies $\beta(s(t)) = \alpha(t)$ where s is the arclength function of α. You may also wish to show that for a unit speed curve β, $\beta'' = k_g(U \times T) + k_n U$.

5. Let α be a regular but not-necessarily unit speed curve in M. Show that

$$\alpha'' = v'T + k_g v^2 (U \times T) + k_n v^2 U$$

where T is the unit tangent to α and $v = |\alpha'|$.

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7. (Practice obtaining numerical solutions to ODEs)
Let $u(t)$ be the solution of the differential equation

$$-hh'' + 1 + h'^2 = ch(1 + h'^2)^{3/2}.$$

a) Use Maple to construct a numerical solution of this differential equation with $h(0) = 1$ and $h'(0) = 0$ (and $c = 0.5$).

b) Make a plot of the solution $h(t)$ for $-20 < t < 20$. Be sure to select 'scaling=constrained' to obtain a good image of the curve.

c) Make a plot of the surface of revolution obtained by revolving this curve about the x-axis. This periodic surface is known as a Delaunay surface and has mean curvature $H = 1/4$ everywhere.