Some review:

In class we defined the Christoffel symbols via the equation

$$x_{\alpha\beta} = \Gamma^u_{\alpha\beta}\mathbf{x}_u + \Gamma^v_{\alpha\beta}\mathbf{x}_v + A_{\alpha\beta}U$$

where α and β are either u or v and

$$[A_{\alpha\beta}] = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

We showed that the Christoffel symbols can be computed by

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^u_{\alpha\beta} \\ \Gamma^v_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{\alpha\beta}, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_{\alpha\beta}, \mathbf{x}_v \rangle \end{pmatrix}.$$

We also computed the right-hand sides of the previous equations:

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2}E_u \qquad \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2}E_v \qquad \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = F_v - \frac{1}{2}G_u$$
$$\langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = F_u - \frac{1}{2}E_v \qquad \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2}G_u \qquad \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2}G_v.$$

From the equation $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, \mathbf{x}_v \rangle = 0$ we computed

$$\begin{aligned} ln - m^2 = {} & \partial_v \left[\langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \mathbf{x}_v \rangle\right] - \partial_u \left[\langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \mathbf{x}_v \rangle\right] \\ & - \langle \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v, \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v \rangle + \langle \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v, \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v \rangle . \end{aligned} \tag{1}$$

Since all quantities on the right-hand side of (1) can be computed from knowledge of E, F, and G alone, we concluded that one can compute Gauss curvature from the first fundamental form.

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1. Without computing anything new, write down an analogous formula to (1) that would be obtained from the equation $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, \mathbf{x}_u \rangle = 0$.

2. Compute a formula analogous to (1) that would be obtained from

$$\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, \mathbf{x}_u \rangle = 0.$$

This one requires actual computation.

3. Show that $\langle (\mathbf{x}_{uu})_v - (\mathbf{x}_{uv})_u, U \rangle = 0$ implies the equation

$$l_v - m_u = \Gamma^u_{uv} l + (\Gamma^u_{uv} - \Gamma^u_{uu})m - \Gamma^v_{uu} n$$

This is one of the two Codazzi-Mainardi equations. The other Codazzi-Mainardi equation comes from $\langle (\mathbf{x}_{vv})_u - (\mathbf{x}_{vu})_v, U \rangle = 0$. For extra credit, write down what this other equation is. (Do no hard work).

4. Let f, g, and h be functions of u and v. Show that

$$\frac{1}{\sqrt{gh}}\frac{\partial}{\partial v}\left(\frac{f_v}{\sqrt{gh}}\right) = \frac{f_{vv}}{gh} - \frac{1}{2}\frac{f_v g_v}{g^2 h} - \frac{1}{2}\frac{f_v h_v}{gh^2}.$$

5. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Compute all the Christoffel symbols for this chart in terms of E and G and their derivatives.

6. Suppose for some chart $\mathbf{x}$ that $F = 0$ everywhere. Use (1) and the previous two problems to show that

$$K = -\frac{1}{2}\frac{1}{\sqrt{EG}}\left[\partial_v\left(\frac{E_v}{\sqrt{EG}}\right) + \partial_u\left(\frac{E_u}{\sqrt{EG}}\right)\right].$$

7. Suppose for some chart $\mathbf{x}$ that $G = 1$ and $F = 0$ everywhere. Show that

$$K = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial v^2}G$$

8. Consider the surface of revolution $\mathbf{x}(u,v) = (f(u)\cos(v), g(u), f(u)\sin(v))$ where $(f')^2 + (g')^2 = 1$. Write down what E, F, and G are for this chart and compute K from E, F, and G.