A complex number is a number of the form $a+ib$ where $i^2 = -1$. The set of complex numbers is denoted by $\mathbb{C}$.

If $z = x + iy$ and $w = u + iv$ then we define

$$\begin{aligned} z + w &= (x+u) + i(y+v) \\ zw &= (xu - yv) + i(yu + xv) \\ \overline{z} &= x - iy. \end{aligned}$$

It is easy to show that $\overline{zw} = \overline{z}\,\overline{w}$ and $\overline{\overline{z}} = z$.

We can identify points in the plane as complex numbers by matching the ordered pair (x, y) with the complex number $x + iy$. In this worksheet you will explore how geometric operations can be expressed in terms of algebraic operations with complex numbers.

1. Write out the formula for $z\overline{z}$. What does this mean geometrically?

2. Given complex numbers z and w, what is the distance squared from z to w?

3. Write out the expression for $z\overline{w}$ in the form $a + ib$ What is the geometric interpretation of a?

4. If $z\overline{w} = a + ib$, show that $a^2 + b^2 = z\overline{z}w\overline{w}$. Can you hypothesize what the geometric interpretation of b is?

5. Suppose ω is a point on the unit circle. Show that $\overline{\omega}$ is a point on the unit circle. What complex number is $\omega\overline{\omega}$?

6. Let $\omega_\theta = \cos(\theta) + i\sin(\theta)$. Let $z = x + iy$. Write $\omega_\theta z$ in the form $a + ib$.

7. Following your text's notation, let r_{cs} be the rotation of angle θ about the origin. What is $r_{cs}(x, y)$?

8. Find a simple function $r_\theta : \mathbb{C} \to \mathbb{C}$ for the isometry of rotation of angle θ about the origin.

9. What isometry is multiplication by i?

10. Find a simple function $f : \mathbb{C} \to \mathbb{C}$ for the isometry of rotation about the angle between the direction u and the x-axis.

11. Find a simple function $t_u : \mathbb{C} \to \mathbb{C}$ for the isometry of translation in the direction u.

12. Find a function $f : \mathbb{C} \to \mathbb{C}$ that represents rotation of angle θ about the point w.

13. Show that every isometry that is of the form of 0 or 2 reflections can be written in the form $f(z) = \omega z + w$ where ω is on the unit circle and w is an arbitrary complex number.

14. Find a function $f : \mathbb{C} \to \mathbb{C}$ that represents reflection about the x axis.

15. Find a function $f : \mathbb{C} \to \mathbb{C}$ that represents reflection about the line in the direction u through the origin.

16. Find a function $f : \mathbb{C} \to \mathbb{C}$ that represents reflection about the line in the direction u through the point w.

17. Find a function $f : \mathbb{C} \to \mathbb{C}$ that represents a glide reflection that translates an amount u about a line passing through w.

18. What is the general form of an isometry that can be written as a product of 1 or 3 reflections?

19. Suppose $f(z) = (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i)z + 3 + 2i$. What is this isometry?

20. If you know an isometry $f(z)$ is a reflection or a glide reflection, how can you quickly determine which type it is?

21. If you know an isometry is a reflection, how can you quickly find a point on the line of reflection?

22. Suppose $f(z) = i\bar{z} + 3 + 2i$. What is this isometry?