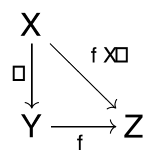
1. Let $\pi: X \to Y$ be a quotient map. Show that if A is a saturated open set or a saturated closed set, then $\pi|_A: A \to \pi(A)$ is a quotient map.

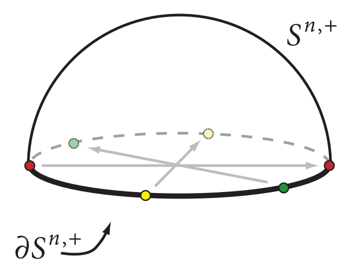
2. (The Characteristic Property of the Quotient Topology is characteristic.) Suppose $\pi: X \to Y$ is a surjective function. Suppose that for any topological space Z that a function $f: Y \to Z$ is continuous if and only if $f \circ \pi$ is. Show that Y has the quotient topology induced by π. (Typeset this one.)



3. Let X be the line with two zeros. That is, X is the quotient space of $\{0,1\} \times \mathbb{R}$ given by the equivalence relation $(0,x) \sim (1,x)$ if $x \neq 0$. Show that X is locally Euclidean and second countable, but not Hausdorff.

4. Suppose X is a second countable space. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in X such that $\bigcup_\alpha U_\alpha = X$. (We call $\{U_\alpha\}_{\alpha \in I}$ an open cover of X.) Prove that there exists a countable sub-collection $\{U_{\alpha_i}\}$ such that $\bigcup_i U_{\alpha_i} = X$. (We say that every open cover of a second countable space has a countable subcover).

5. Use Problem 1 to show that the upper half sphere $S^{n,+}$ with antipodal points on $\partial S^{n,+}$ identified is homeomorphic to $\mathbb{RP}^n$.



Gluing a half sphere.

6. Suppose π is a quotient map from a second countable space X to a locally Euclidean space Y. Prove that Y is second countable. Hint: Problem 4 might be handy.

7. (Munkres 17.7) Let $\mathbb{R}^\infty$ be the set of real sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in the product and box topologies of $\mathbb{R}^\omega$?