Math F608: Midterm Exam **Due: 5:00 November 6, 2006**

Please see the rules on the second page.

1. Let $U = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$. Find a solution of the following problem:

$$\begin{aligned} u_{tt} &= \Delta\, u \quad \text{in } U \times (0,\infty) \\ u(x,y,z,0) &= 0 \\ u_t(x,y,z,0) &= h(x,y,z) \\ u_z(x,y,0,t) &= 0. \end{aligned}$$

2. Let U be a bounded domain with a smooth boundary. Consider the problem

$$\begin{aligned} u_t - \Delta\, u &= f \quad \text{in } U \\ u(x,0) &= g(x) \\ (\partial_\nu + \alpha)u &= 0 \quad \text{on } \partial U \end{aligned}$$

where α is a smooth function on the boundary. This boundary condition is known as a Robin condition (unless $\alpha \equiv 0$, in which case it is called a Neumann condition).

a. Use an energy argument to show that if $\alpha \geq 0$, then there can be at most one $C^2(U) \cap C(\overline{U})$ solution of this problem.

b. Derive a weak maximum principle for this problem.

b. Find a counterexample to show that there can be multiple solutions of this problem if $\alpha < 0$.

c. Use separation of variables to find an eigenfunction expansion solution of this problem on the domain $Q = [0,1] \times [0,1]$ when $\alpha = 0$. Also, compute $\lim_{t\to\infty} u(x,t)$. You may assume as much smoothness from f and g as you wish. While you do not need to be completely formal with arguments involving convergence, do point out where in your argument convergence issues appear.

3. Let U be the region exterior to the unit ball in $\mathbb{R}^n$ with $n \geq 2$. Consider the problem

$$\begin{aligned} \Delta\, u &= 0 \quad \text{in } U \\ u &= f \quad \text{on } \partial U. \end{aligned}$$

a. Show that for all dimensions $n \geq 2$ there is not a unique solution to this problem.

b. Show that if one adds the condition that $\lim_{x\to\infty} u(x) = 0$ that there can exist no more than one solution of this problem.

c. Find a representation formula for the solution of this problem, with the additional condition from subproblem **b** assumed.

4. Find the general solution of the equation

$$x u_{xx} + u_{xy} = 0.$$

5. Consider the following *non-linear* problem

$$\begin{aligned} -\Delta\, u + u^3 &= 1 \quad \text{in } U \\ u &= 0 \quad \text{on } \partial U \end{aligned}$$

on a smooth bounded domain U in $\mathbb{R}^n$.

a. Formulate (and prove) a variational principle (analagous to Dirichlet's principle) for a $C^2(U) \cap C(\overline{U})$ solution of this problem.

b. Show that there can exist at most one such solution of this problem.

6. Consider Laplace's equation in one dimension.

a. Find a fundamental solution for the Laplacian.

b. Suppose f is a Riemann integrable function in $L^1(\mathbb{R})$. Use your answer from part **a** to find (with proof) a weak solution of

$$-u'' = f.$$

c. Show that your solution in part **b** is continuous.

d. Show that if f is bounded, then your solution from part **b** is in $C^1(\mathbb{R})$.

7.

a Solve the PDE

$$u_x + u^2 u_y = 1$$

with initial condition $u(x, 0) = 1$.

b Solve the PDE

$$u_x + \sqrt{u} u_y = 0$$

with initial condition $u(x, 0) = 1 + x^2$.

Rules and format:

1. You are welcome to discuss this exam with me (David Maxwell) to ask for hints and so forth.
2. You my not discuss the exam with anyone else until after the due date/time.
3. You are permitted to use Evans or any other PDE text you like. If you use another text, you must cite it when you use it.
4. Each problem is weighted equally.
5. The due date/time is absolutely firm.
6. If requested, there will be a hints session held at a time to be arranged later.