Math F608: Homework 6 **Due: October 16**

1. Consider Cauchy problem for Laplace's equation on the square $Q = [0,1] \times [0,1]$,

$$\begin{aligned} \Delta u &= 0 \quad \text{in } Q \\ u(x,y) &= 0 \quad \text{for } x = 0 \text{ or } x = 1 \text{ and } y \in [0,1]. \\ u(x,0) &= g(x) \\ u_y(x,0) &= h(x). \end{aligned}$$

For simplicity we will consider the case $h(x) \equiv 0$.

a. Use a separation of variables technique to find a formal solution of this problem. You do not need to be rigorous about convergence issues.

b. Find a sequence of functions $g_n(x)$ that converge uniformly to 0 but such that the corresponding solution $u_n(x)$ to the Cauchy problem satisfies $\sup_x |u(x,1)| \geq 1$.

c. Comment on the well posedness of the Cauchy problem. (This example is due to Hadamard.)

2. Let U be a smooth bounded domain, and let $\{\phi_m\}$ be the Dirichlet eigenfunctions of the Laplacian with eigenvalues $\{\lambda_n\}$ (so $-\Delta\, \phi_n = \lambda_n \phi_n$ and $\phi_n = 0$ on ∂U).

a. Prove that $\lambda_n \geq 0$ for every n.

b. Prove that if $\lambda_n \neq \lambda_m$, then $\int_U \phi_n \phi_m = 0$.

3. Suppose a square plate with side length 1 has a constant temperature $U_0 > 0$ and at time $t = 0$, the temperature at the edge of the plate is dropped to 0. Formally apply separation of variables to this problem and describe the resulting diffusion.

4. Let U be a smooth bounded domain and suppose $f(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$. Find a formal solution by means of eigenfunction expansion to the problem

$$\begin{aligned} u_t - \Delta\, u &= f \quad \text{in } U \\ u &= 0 \quad \text{at } t = 0 \text{ and on } \partial U. \end{aligned}$$

5. Let λ_0 be the smallest eigenvalue of the laplacian on a bounded smooth domain U.

a. Using eigenfunction expansions, give a heuristic argument that

$$\lambda_0 = \min_{\substack{u \in C^2(U) \cap C(\overline{U}) \\ u \not\equiv 0}} \frac{\int_U |Du|^2}{\int_U u^2}.$$

The quotient appearing this this problem is called the Rayleigh quotient.

b. Suppose $u(x,t)$ solves

$$\begin{aligned} u_t - \Delta\, u &= 0 \quad \text{in } U \times (0,\infty) \\ u &= 0 \quad \text{on } \partial U \times (0,\infty). \end{aligned}$$

Use the result from part **a**, and letting

$$e(t) = \int_U u(x,t)^2 \, dx,$$

show that

$$e(t) \leq e(0) e^{-\lambda_0 t}.$$

You may not use informal eigenfunction expansions for this part of the problem.