1.

a. Let U be an open subset of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be analytic (i.e complex differentiable, or holomorphic). Show that the real and imaginary parts of f thought of as functions from $\mathbb{R}^2$ to $\mathbb{R}$ are both harmonic functions.

b. Let U_θ be the domain $\{(r\cos(s), r\sin(s)) \in \mathbb{R}^2 : 0 < r < \infty, 0 < s < \theta\}$. For example, U_π is the upper half plane, and $U_{\pi/2}$ is the first quadrant. Find functions (not identically 0) u_θ that solve

$$\begin{aligned} \Delta\, u_\theta &= 0 \quad \text{in } U_\theta \\ u_\theta &= 0 \quad \text{on } \partial U_\theta. \end{aligned}$$

Hint: For the domain $U_{\pi/2}$, is there an analytic function that takes $U_{\pi/2}$ to U_π?

c. Discuss the smoothness of the functions you found in part **b.** at the point $(0,0)$.

2. Find a Greens function for the domain $U_{\pi/2}$. (Note that we are working in two dimensions, so the fundamental solution is $-1/\pi \ln(r)$.

3. Let U be a smooth domain, and let f be a continuous function on U. Suppose u is a solution of

$$\begin{aligned} -\Delta\, u &= f \quad \text{in } U \\ \frac{\partial}{\partial \nu} u &= 0 \quad \text{in } U. \end{aligned}$$

Show that u is a minimum of the functional

$$I[w] = \int_U |Dw|^2 - fw$$

over the domain $\mathcal{B} = \{w \in C^2(\overline{U})\}$.

4. Evans 2.5.5

5. Evans 2.5.9

Howdy.