1. This problem gives a way of computing Fourier transforms of some functions that are distributions but do not live in L^1 because they decay too slowly.

a. Suppose $f \in L^1_{\text{loc}}(\mathbb{R})$ and that there exist a constant $N > 0$ such that $(1+|x|)^{-N} f \in L^1(\mathbb{R})$. It is easy to see that such an f belongs to $\mathcal{S}'$. Let L_n be any sequence of positive real numbers converging to infinity, and let χ_n be functions that are equal to 1 on $[-L_n, L_n]$, and are 0 elsewhere. Let g_n denote the Fourier transform of $\chi_n f$. Show that $g_n \to \hat{f}$ in $\mathcal{S}'$.

b. Let $f \in L^1(\mathbb{R})$. We proved in class that $\hat{f}$ is continuous, and we claimed that $\hat{f}(\xi)$ tends to 0 as $\xi \to \pm\infty$. Prove this claim. You may freely use the fact that $\mathcal{S}$ is dense in $L^1(\mathbb{R})$. *Hint:* If $u \in L^1$ and $v \in \mathcal{S}$, and if $||u - v||_{L^1} < \epsilon$, what can be said about $||\hat{u} - \hat{v}||_{L^\infty}$? What space does $\hat{v}$ belong to?

c. Let L_n be a sequence of positive real numbers converging to ∞. Prove that the distributions $e^{-2\pi i L_n \xi}$ converge to 0 in $\mathcal{S}'$ as $n \to \infty$.

d. Use these ideas to compute the Fourier transform of $\operatorname{sgn}(x)$.

2.

a. Two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space X are called equivalent if there exist constants m and M such that for every x in X,

$$m||x||_1 \le ||x||_2 \le M||x||_1.$$

Show that on $\mathbb{R}^n$ that the Euclidean norm defined by

$$||x|| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}$$

and the L^1 norm defined by

$$||x||_1 = \sum_{i=1}^n |x_i|$$

are equivalent.

b. Show that for any $k \in \mathbb{N}$, there exists constants c_1 and c_2 such that

$$c_1(1 + |x|^k) \le (1 + |x|^2)^{k/2} \le c_2(1 + |x|^k).$$

c. Conclude that if $u \in L^2$, then $u \in H^k$ if an only if $\partial^\alpha u \in L^2$ for every α with $|\alpha| = k$.

3. Suppose $u \in H^s$. Show that if $|\alpha| = k$, then $\partial^\alpha u \in H^{s-k}$. Show also that $\partial^\alpha : H^s \to H^{s-k}$ is continuous.

4. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the function that is equal to 1 on the unit square and is equal to 0 otherwise. Which spaces H^s does u belong to?