1.

a. Let $1/x$ denote the Cauchy Principle Value distribution defined by

$$\langle 1/x, v\rangle = \lim_{\epsilon\to 0}\int_{-\infty,-\epsilon}\frac{v(x)}{x}\,dx + \int_{\epsilon,\infty}\frac{v(x)}{x}\,dx$$

a. Show that $1/x$ is well defined and is a distribution.

b. Prove that $\partial_x \ln(|x|) = 1/x$ in the sense of distributions. You may use the well known fact that $\ln(|x|) \in L^1_{\text{loc}}(\mathbb{R})$.

2. Show that $\langle F, v\rangle = \sum_{k=1}^{\infty} v^(k)(1/k)$ is a distribution on $(0,\infty)$ but not on $\mathbb{R}$.

3. Show that if $xF = 0$ for some distribution F, then $F = c\delta$ for some real number c. *Hint:* Try to write $v \in \mathcal{D}(\mathbb{R})$ as $v(0)\eta(x) + x\phi_v(x)$ for smooth functions ϕ_v and η, where η does not depend on v.

4. We say that a sequence of distributions $\{F_n\}$ in $\mathcal{D}'(U)$ converges (weakly) to F if for every $v \in \mathcal{D}(U)$, $\langle F_n, v\rangle \to \langle F, v\rangle$. Show that the step functions f_n that are equal to n on $[0, 1/n]$ and are otherwise 0 converge to δ.

5. Consider the map $\tau_h : L^1_{\text{loc}}(\mathbb{R}) \to L^1_{\text{loc}}(\mathbb{R})$ given by

$$\tau_h(f)(x) = \frac{f(x+h) - f(x)}{h}$$

a. Give a definition of τ_h that extends it to a map from $\mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$.

b. Show that for any sequence $\{h_n\}$ converging to 0, and for any distribution F, that $\tau_{h_n} F$ converges to F'.

c. (Requires Functional Analysis) Suppose $f \in L^2(\mathbb{R})$ and $||\tau_h f||_{L^2(\mathbb{R})}$ is bounded above by M for every h. Prove that the distributional derivative of f is also an element of $L^2(\mathbb{R})$.