1.

a. Show that S_Ω does not have a largest element.

b. Show that if $\alpha \in S_\Omega$, then either α has an immediate predecessor (i.e. $\alpha = \beta^+$ for some β), or there exists a countable increasing sequence of ordinal numbers $\{\alpha_i\}$ such that α is the least upper bound of the sequence.

c. Show that the set of elements in S_Ω that do not have an immediate predecessor is uncountable.

2.

(Based on Munkres 24.12) For this problem, you can freely use the fact that if $f : X \to Y$ is an order isomorphism of partially ordered spaces, then it is a homeomorphism when these spaces are given the order topology.

a. Let X be a linearly ordered set and let $x_0 < x_1 < \cdots$ be a sequence of points in X such that each interval $[x_i, x_{i+1})$ is order isomorphic to $[0, 1)$. Show that $\bigcup_{i=1}^{\infty}[x_i, x_{i+1})$ is order isomorphic to $[0, 1)$.

b. Consider the long closed ray $\overline{L^+} = S_\Omega \times [0, 1)$. Show that for each $\alpha \in S_\Omega$ not equal to the initial element α_0, the interval $I_\alpha = \{x \in \overline{L^+} : x < (\alpha, 0)\}$ is order isomorphic to $[0, 1)$. *Hint:* Use transfinite induction (and problem **1b** in the case that α does not have an immediate predecessor.)

c. Show that for every $x, y \in \overline{L^+}$ with $x < y$, the interval $[x, y) \subset \overline{L^+}$ is homeomorphic to $[0, 1) \subset \mathbb{R}$ and the interval (x, y) is homeomorphic to $(0, 1) \subset \mathbb{R}$. (*Hint:* Consider the case x is equal to the initial element first.)

d. Show that L^+ is Hausdorff, locally Euclidean, but **not** second countable. Such objects are considered to be manifolds by some authors and are called **non-paracompact manifolds**.

3.

Show that $\overline{L^+}$ is sequentially compact but not compact. *Hint:* If $\{x_n\}$ is a sequence in $\overline{L^+}$, consider the map taking n to the ordinal α_n such that $x_n \in \alpha_n \times [0, 1)$. Also show that if $\{x_n\}$ is a non-decreasing sequence in $\overline{L^+}$, then it converges.

4.

Let $\overline{S_\Omega} = \Omega^+$. That is, $\overline{S_\Omega} = S_\Omega \cup \{\Omega\}$ and $\Omega \geq x$ for all $x \in S_\Omega$. Determine whether each of S_Ω and $\overline{S_\Omega}$ are first countable countable, second countable, or neither. Justify your answers.

5.

(Munkres 11.8) Let V be a vector space. Recall that if $W \subset V$, the set $\mathrm{span}(W)$ is the collection of all finite linear combinations of vectors in W. Recall also that W is **linearly independent** if for each $w \in W$, $w \notin \mathrm{span}(W - \{w\})$.

a. Show that if W is a linear independent subset of V and if v is a vector not in $\mathrm{span}(W)$, then $W \cup \{v\}$ is a linearly independent set.

b. Order the set $\mathcal{W}$ of linearly independent subsets of V by inclusion. Use Zorn's Lemma to show that $\mathcal{W}$ has a maximal element.

c. Show that V has a basis (i.e. a linearly independent set W such that $\mathrm{span}(W) = V$).