Math F651: Homework 13 **Due: May 2**

1: Munkres 53.4.

2: Munkres 59.3.

3: The Brower fixed point theorem in dimension 2. Let $\mathbb{B}^2$ be the closed unit ball in $\mathbb{R}^2$. Show that for any continuous function $F : \mathbb{B}^2 \to \mathbb{B}^2$, there is a point $x \in \mathbb{B}^2$ such that $F(x) = x$.

Hint: Suppose there is no fixed point and consider the map $G : S^1 \to S^1$ given by $G(x) = \frac{x-F(x)}{|x-F(x)|}$. Show that G is homotopic to the identity map on S^1 and also that G is null homotopic to derive a contradiction. Note: Your book has a different, longer proof of this result. You might find Lemma 55.3 useful in your proof.

4.

a. Let X be a topological space and let $H : I \times I \to X$ be continuous. Define paths

$$\begin{aligned} f(t) &= H(0,t) \\ g(t) &= H(t,1) \\ h(t) &= H(t,0) \\ k(t) &= H(1,t) \end{aligned}$$

show that $f \cdot g \sim h \cdot k$.

b. Show that the fundamental group of any topological group G at any base point is abelian. *Hint:* Given loops α and β in G based at the idenity, consider the map $H : I \times I \to G$ defined by $H(s,t) = \alpha(s)\beta(t)$.

5. Let $d : \pi_1(S^1, 1) \to \mathbb{Z}$ be the isomorphism we defined in class given by

$$d([f]) = \tilde{f}(1)$$

where $\tilde{f}$ is the lift of f such that $\tilde{f}(0) = 0$. For any map $F : S^1 \to S^1$ we define the **degree** of F via

$$\deg(F) = d(\hat{F}_*([\epsilon]))$$

where $\epsilon(t) = e^{2\pi i t}$ and where $\hat{F}(z) = F(z)/F(1)$.

a. Show that two maps F and G from S^1 to S^1 are homotopic if and only if they have the same degree.

b. Show that $\deg(F \circ G) = \deg(F)\deg(G)$ for maps F, G from S^1 to S^1.

6. The Klein bottle is $\mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n,m) \cdot (x,y) = (x+n, (-1)^n y + m)$. Find a two sheeted covering of the Klein bottle by the torus.