EXTREMAL PROBLEMS IN GRAPH THEORY

BY

CHRISTOPHER M. HARTMAN

B.S., University of Alaska, Fairbanks, 1991

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1997

Urbana, Illinois

Abstract

We consider generalized graph coloring and several other extremal problems in graph theory. In classical coloring theory, we color the vertices (resp. edges) of a graph requiring only that no two adjacent vertices (resp. incident edges) receive the same color. Here we consider both weakenings and strengthenings of those requirements. We also construct twisted hypercubes of small radius and find the domination number of the Kneser graph $K(n,k)$ when $n \geq \frac{3}{4}k^2 + k$ if k is even, and when $n \geq \frac{3}{4}k^2 - k - \frac{1}{4}$ when k is odd.

The *path chromatic number* $\chi_P(G)$ of a graph G is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of paths. We answer some questions of Weaver and West [31] by characterizing cartesian products of cycles with path chromatic number 2.

We show that if G is a toroidal graph, then for any non-contractible chordless cycle C of G, there is a 3-coloring of the vertices of G so that each color class except one induces a disjoint union of paths, while the third color class induces a disjoint union of paths and the cycle C.

The *path list chromatic number* of a graph, $\hat{\chi}_P(G)$, is the minimum k for which, given any assignment of lists of size k to each vertex, G can be colored by assigning each vertex a color from its list so that each color class induces a disjoint union of paths. We strengthen the theorem of Poh [24] and Goddard [11] that $\chi_P(G) \leq 3$ for each planar graph G by proving also that $\hat{\chi}_P(G) \leq 3$.

The *observability* of a graph G is least number of colors in a proper edge-coloring of G such that the color sets at vertices of G (sets of colors of their incident edges) are pairwise distinct. We introduce a generalization of observability. A graph G has a *set-balanced*

k-edge-coloring if the edges of G can be properly colored with k colors so that, for each degree, the color sets at vertices of that degree occur with multiplicities differing by at most one. We determine the values of k such that G has a set-balanced k-edge-coloring whenever G is a wheel, clique, path, cycle, or complete equipartite multipartite graph. We prove that certain 2-regular graphs with n vertices have observability achieving the trivial lower bound $\min\{j : \binom{j}{2} \geq n\}$. Horňák conjectured that this is always so.

The *spot-chromatic number* of a graph, $\chi_S(G)$, is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of cliques. We generalize a construction of Jacobson to show that $\chi_S(K_{mt} \Box K_{nt}) \leq \frac{mnt}{m+n} + 2\min(m, n)$ whenever $m + n$ divides t. The construction is nearly optimal.

Twisted hypercubes, generalizing the usual notion of hypercube, are defined recursively. Let $\mathcal{G}_0 = \{K_1\}$. For $k \geq 1$, the family $\mathcal{G}_k$ of *twisted hypercubes* of *dimension* k is the set of graphs constructible by adding a matching joining two graphs in $\mathcal{G}_{k-1}$. We construct a family of twisted hypercubes of small diameter. In particular, we prove that the order of growth of the minimum diameter among twisted hypercubes of dimension k is $\Theta(k/\lg k)$.

The domination number $\gamma(G)$ of a graph G is the minimum size of a set S such that every vertex of G is in S or is adjacent to some vertex in S. The Kneser graph $K(n, k)$ has as vertices the k-subsets of $[n]$. Two vertices of $K(n, k)$ are adjacent if the k-subsets are disjoint. We determine $\gamma(K(n, k))$ when $n \geq \frac{3}{4}k^2 \pm k$ depending on the parity of k.

Dedication

To Heather and Sarah.

Acknowledgement

I would like to thank Glenn Chappell and the other combinatorics students at the University of Illinois for many hours of helpful discussion.

Of course, my advisor, Doug West, deserves great thanks. Without his teaching I wouldn't have discovered the beauty of graph theory, and without his guidance and advice I wouldn't have made it through graduate school.

I would also like to thank George Francis for his advice and general helpfulness.

Finally, I'd like to thank my wife and my parents for their support and encouragement.

Table of Contents

1. Introduction **1**

1.1. Generalized graph coloring **1**

1.1.1. Path coloring **3**

1.1.2. Edge-coloring **7**

1.2. Other extremal problems **8**

1.2.1. Twisted Hypercubes **8**

1.2.2. The domination number of the Kneser graph **9**

1.3. Additional notation and terminology **10**

2. Path colorings of cartesian products of cycles **15**

2.1. Products of three or fewer odd cycles **16**

2.2. Products of four or more odd cycles **21**

3. Path colorings of toroidal graphs **24**

3.1. Path coloring of planar graphs **24**

3.2. Coloring torus graphs **27**

3.3. Other Results **34**

4. Path list colorings of planar graphs **37**

5. Spot colorings **44**

5.1. Spot coloring the cartesian product of cliques **44**

5.2. Jacobson's construction **46**

5.3. The construction for $K_m \,\square\, K_n$ **47**

6. Observability and set-balanced k-edge-coloring **51**

6.1. Special classes **51**

6.2. Observability of regular graphs **61**

7. Twisted Hypercubes **67**

7.1. The construction **70**

8. Domination number of Kneser graphs **75**

8.1. Monotonicity of $\gamma(n,k)$ **76**

8.2. The upper bound **77**

8.3. The lower bound **79**

8.4. Constructions for smaller n. **82**

References **85**

Vita **87**

1. Introduction

Extremal graph problems often involve finding the extreme value of a parameter over some class of graphs. Here we study several such problems involving generalized coloring parameters, diameter, and domination number. We include algorithms to produce generalized colorings of planar and toroidal graphs and constructions for parameters on specific graphs that we prove are optimal or nearly optimal.

1.1. Generalized graph coloring

Classical coloring theory can be viewed as modeling conflict avoidance. The edges of a graph represent conflicts among the vertices, and we seek a partition of vertices into conflict-free classes. We do this by assigning labels called *colors*, requiring only that no two adjacent vertices receive the same color. If this can be done with k colors, then we say that the graph is *k-colorable*. The minimum number of colors required to color the vertices of a graph G in this way is the *chromatic number*, $\chi(G)$.

We can also color the edges of a graph. In classical coloring theory, we require only that no two incident edges receive the same color, that is, that each color class is a matching. A graph is *k-edge-colorable* if it can be colored in this way with k colors. The *edge-chromatic number*, $\chi'(G)$, is the minimum k such that G is k-edge-colorable.

Generalizations of graph coloring consider both weakenings and strengthenings of these requirements. When weakening the requirement for vertex coloring, we usually define a graph property or graph family Q and let $\chi_Q(G)$ be the minimum number of colors required to color the vertices of G so that the subgraph induced by each color class has property Q or belongs to family Q. Such a coloring using i colors is a *Q i-coloring*. (In the ordinary or generalized setting, an i-coloring is an arbitrary labeling of the vertices from a set of i colors. In ordinary coloring we prepend the word "proper" to impose the desired constraint; in generalized coloring we prepend the designation of the property.) In this dissertation we consider the properties $P = \{H : H$ is a disjoint union of paths$\}$, $P_k = \{H : H$ is a disjoint union of paths each having no more than k vertices$\}$ and $S = \{H : H$ is a disjoint union of cliques$\}$. If all graphs with property Q have property Q', then $\chi_Q(G) \geq \chi_{Q'}(G)$, since every Q i-coloring of G is also a Q' i-coloring of G. Thus $\chi_P(G) \leq \cdots \leq \chi_{P_3}(G) \leq \chi_{P_2}(G) \leq \chi_{P_1}(G) = \chi(G)$ and $\chi_S(G) \leq \chi(G)$.

One way to strengthen the vertex coloring requirement is *list coloring*. When list coloring a graph, we must choose the color for each vertex from a list specified for that vertex. A graph is *k-choosable* if it is properly colorable in this manner whenever the lists chosen for the vertices have size k. The list chromatic number of a graph, $\hat{\chi}(G)$, is the minimum k such that G is k-choosable. Determining whether $\chi(G) \leq k$ is equivalent to the specific instance of list coloring where each vertex is given the list $\{1, 2, \ldots, k\}$; thus $\chi(G) \leq \hat{\chi}(G)$.

List coloring is often used to model coloring problems where some vertices have been pre-colored. We can take those vertices out of such a graph and remove their colors from neighboring lists. Solving the list coloring problem for such a graph results in a proper coloring of the original graph. One application of this is the assignment of radio frequncies to a new set of transmitters. Conflicts with existing transmitters must be avoided, as well as conflicts between new transmitters. List coloring also can provide a way to make an efficient coloring algorithm. An algorithm can keep track of which colors are unavailable at a vertex (because some neighbor already has that color) using list coloring. One example is Thomassen's proof [29] that planar graphs are 5-choosable. This proof leads directly to the first linear-time algorithm for 5-coloring a planar graph. Voigt [30] provided an example with 238 vertices showing that there are planar graphs that are not 4-choosable. A simpler example, with only 63 vertices, was found by Mirzakhani [20].

Here we combine list coloring with the weakened coloring requirement that each color class induce a disjoint union of paths. We define the *path list chromatic number* $\hat{\chi}_P(G)$ to be the minimum k such that if each vertex of G is assigned a list of size k, then G can be colored so that each vertex receives a color from its list, and each color class induces a disjoint union of paths.

We also consider restricted edge-colorings in graphs. Note that when coloring vertices we speak of an i-coloring, but when coloring edges we specify an i-edge-coloring. We place additional restrictions on the sets of colors on edges incident to a vertex. Ordinary edge coloring requires only that no color appears twice among the edges at a vertex. We require additionally that no *set* of incident colors appears at more than one vertex. Further, we may require that each such set of size d appears at $\lfloor n_d/\binom{k}{d} \rfloor$ or $\lceil n_d/\binom{k}{d} \rceil$ vertices, where n_d is the number of vertices of degree d.

1.1.1. Path coloring

The *path chromatic number* $\chi_P(G)$ of a graph G is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of paths. ("Disjoint union of paths" means "maximum degree 2 and no cycles.") Such a coloring is a *path coloring*. We say that G is *path i-colorable* if $\chi_P(G) \le i$ and that G is *path i-chromatic* if $\chi_P(G) = i$. This extends the terminology of ordinary graph coloring by prepending the designation of the generalized property.

In this dissertation we study path coloring for two types of graphs. First, we consider cartesian products of cycles. In graph theory we often wish to know how the cartesian product operation affects graph invariants. For example, $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$ is an elementary observation about cartesian products and the classical chromatic number.

In Chapter 2 we determine $\chi_P(G)$ when G is a cartesian product of cycles. As observed by Weaver and West [31], $\chi_P(G)$ depends only on the odd factors of G. Since $\chi(G) \leq 3$ when G is a cartesian product of cycles, we always have $\chi_P(G) \leq 3$. We characterize which cartesian products of odd cycles G satisfy $\chi_P(G) = 2$. We also consider $\chi_{P_k}(G)$, the generalized chromatic number obtained when we require that each color class induces a disjoint union of paths each having at most k vertices. For products of two cycles, Weaver and West determined the values of k (depending on the lengths of the cycles) such that $\chi_{P_k}(G) = 2$. We show that $\chi_P(G) = 3$ whenever G contains the product of four or more odd cycles. Since $\chi_{P_k}(G) \geq \chi_P(G)$, this also implies that $\chi_{P_k}(G) = 3$ for all k and all such G. For products of exactly three odd cycles, we show that the minimal triples (a, b, c) such that $C_a \,\square\, C_b \,\square\, C_c$ is path 2-colorable are $(5, 7, 7)$ and $(5, 5, 11)$. These triples are minimal in the sense that increasing the (odd) length of any component results in a path 2-colorable graph, but decreasing any component leaves a graph that is not path 2-colorable. Note that every triple is comparable to at least one of these. We also prove that $\chi_{P_4}(G) = 3$ for all products of three odd cycles, and we prove that $\chi_P(C_{15} \,\square\, C_{15} \,\square\, C_{15}) = \chi_{P_5}(C_{15} \,\square\, C_{15} \,\square\, C_{15}) = 2$. This implies the same results for all larger products of three odd cycles. Additionally, we obtain partial results (with the aid of a computer) on how large k must be to obtain $\chi_{P_k}(C_a \,\square\, C_b \,\square\, C_c) = 2$ when a, b, and c are each at most 15.

Coloring theory was first popularized by the four color problem: Is every planar graph

4-colorable? This problem lead Heawood [12] to determine an upper bound on the chromatic number for graphs on a surface. The Heawood bound depends only on ε, the Euler characteristic of the surface, but it is valid only for $\varepsilon < 2$, that is, for surfaces other than the plane. Franklin [10] improved Heawood's bound by one for graphs on the Klein bottle. Ringel and Youngs [25] eventually completed the solution for surfaces other than the plane by proving that Heawood's upper bound is attained by cliques. In 1977 Appel, Haken, and Koch, [2] [3] proved the Four Color Theorem, stating that planar graphs are 4-colorable. This was the final link to solve the problem of determining the maximum chromatic number of graphs embedded on surfaces. In Chapters 3 and 4 we continue the long tradition of examining coloring problems for graphs on surfaces by examining path coloring of graphs on the plane and torus.

Akiyama, Era, Gervacio, and Watanabe [1] were the first to consider $\chi_P(G)$ for various planar graphs. They proved that outerplanar graphs are path 2-colorable and that for all k there are planar graphs that are not P_k 3-colorable. They conjectured that $\chi_P(G) \leq 3$ for planar graphs. This conjecture was proven independently by Poh [24] and by Goddard [11].

Figure 1.1 shows a path 3-chromatic planar graph. If we were to color it with only two colors, then each of the three triangles that don't include the central vertex would require both colors (otherwise we would have a monochromatic cycle), but then the central vertex would have at least three neighbors of its own color, no matter which color it receives.

Cowen, Goddard, and Jesurum [7] considered a property weaker than P, requiring only that each color class induce a graph with maximum degree 2. This allows both paths and cycles as components of the subgraph induced by a color class. They proved that toroidal

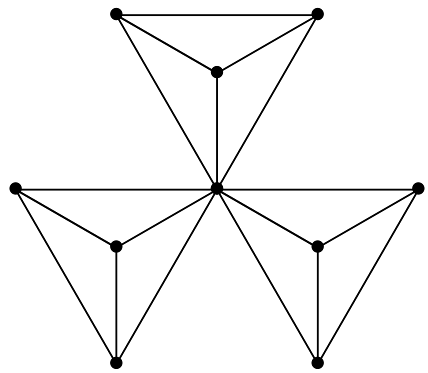

Figure 1.1

graphs are 3-colorable with respect to this property. We strengthen this result by showing that if G is a toroidal graph, then for every noncontractible chordless cycle C of G there is a 3-coloring of the vertices of G so that two color classes induce a disjoint union of paths, while the third color class induces a disjoint unions of paths and the cycle C. Since removing one edge from a cycle leaves a path, we conclude that if G is a toroidal graph then there exists an edge e of G such that $\chi_P(G - e) \leq 3$. The clique K_7 embeds on the torus and shows that this is best possible: if we use only three colors on seven vertices we must have three vertices with the same color, which yields a monochromatic cycle.

As a corollary to our toroidal result, we conclude immediately that toroidal graphs can always be 7-colored with some color class containing at most one vertex.

In Chapter 4 we study $\hat{\chi}_P(G)$, the path list chromatic number, for planar graphs. Such a problem might arise, for example, if we need to assign radio frequencies to transmitters where interference is acceptable along a path, but in no larger subgraph. The lists could in this case model restrictions on the frequencies that the transmitters could produce, or restrictions due to pre-existing transmitters, or both. While this particular example may be a bit far-fetched, it nevertheless shows the nature of real-world applications of path list coloring.

We show that the path list chromatic number is at most 3 for all planar graphs. (Recall that $\hat{\chi}_P(G) \leq k$ if, for any assignment of lists of size k to each vertex, G can be path colored by assigning each vertex a color from its list.) This strengthens the theorem of Poh and Goddard.

The *spot chromatic number* of a graph, $\chi_S(G)$, is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of cliques. Jacobson [13] showed that $\chi_S(K_t \square K_t) = \frac{t}{2} + 2$ when t is even. In Chapter 5 we generalize his construction to show that $\chi_S(K_{mt} \square K_{nt}) \leq \frac{mnt}{m+n} + 2\min(m,n)$ whenever $m+n$ divides t. This construction is asymptotically optimal.

1.1.2. Edge-coloring

The *observability* of a graph G is the least number of colors in a proper edge-coloring of G such that the color sets at vertices of G (sets of colors of their incident edges) are pairwise distinct. This concept was introduced by Černý, Horňák and Soták [5]. In Chapter 6 we consider a generalization of the observability of a graph. A *set-balanced k-edge-coloring* of a graph G is a proper k-edge-coloring of G such that for each vertex degree d each d-set of colors appears at $\lfloor n_d/\binom{k}{d} \rfloor$ or $\lceil n_d/\binom{k}{d} \rceil$ vertices, where n_d is the number of vertices of degree d. For example, a d-regular graph G with n vertices has a set-balanced k-edge-coloring if the edges of G can be properly colored so that each set of d colors appears at $\lfloor n/\binom{k}{d} \rfloor$ or $\lceil n/\binom{k}{d} \rceil$ vertices. Thus, when k is large enough, the question of whether G has a set-balanced k-edge-coloring becomes the question of whether the observability of G is at most k.

Černý, *et. al* determined the observability of complete graphs, paths, cycles, wheels, and complete multipartite graphs with partite sets of equal size. We generalize these results to state exactly the values of k such that graphs from these classes have set-balanced k-edge-colorings. We also prove that certain 2-regular graphs with n vertices have observability equal to the trivial lower bound $\min\{j : \binom{j}{2} \geq n\}$. Horňák conjectured that this holds for all 2-regular graphs. This conjecture is equivalent to "Given a set of lengths summing to $\binom{n}{2}$, the edges of K_n can be partitioned into closed trails realizing those lengths."

1.2. Other extremal problems

1.2.1. Twisted Hypercubes

In multi-processor computers, processors must be able to communicate to work together on a problem. However, it is often impractical to have every processor able to communicate directly with every other processor. We can model a communications network using a graph with a vertex for each processor and an edge between processors if they can communicate with each other directly. Limiting the number of different processors a processor can communicate with corresponds to limiting the maximum degree in the graph. The k-dimensional hypercube Q_k is a typical communication graph; it describes 2^k processors, each able to communicate directly with k others. The diameter of a communication graph represents the maximum number of steps a message would have take to get from one processor to another. This can be viewed as a measure of the maximum delay for message transmission. The diameter of Q_k is k. Many people have studied hypercube variants in order to produce a graph similar to the hypercube but with smaller diameter.

Larson and Cull [17] list various hypercube-like graphs with 2^k vertices and diameter $\frac{k}{2}$. They improve upon these results by constructing certain "twisted" hypercubes with 2^k vertices and diameter $\frac{2k}{5}$. *Twisted hypercubes*, which generalize the usual notion of hypercube, are defined recursively. Let $\mathcal{G}_0 = \{K_1\}$. For $k \geq 1$, let $\mathcal{G}_k$ denote the set of graphs constructible by adding a matching joining two graphs in $\mathcal{G}_{k-1}$. Here k is the *dimension* of the resulting twisted hypercube. In Chapter 7 we construct twisted hypercubes of dimension k with diameter on the order of $4\frac{k}{\lg k}$. This is the first construction of twisted hypercubes with sublinear diameter, and its diameter has the same order as the lower bound.

1.2.2. The domination number of the Kneser graph

A *dominating set* S of a graph G is a a set $S \subseteq V(G)$ such that every vertex of $V(G) \setminus S$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of a graph G is the minimum size of such a set. Finding the domination number of a graph has applications in computer networking. If a computer network is represented as a graph, the domination number represents the minimum number of servers needed to ensure that each computer is adjacent to a server. Note that $\gamma(G)$ can be thought of as a generalized coloring parameter. It is the minimum number of colors with which G can be colored so that each color class induces a subgraph with a dominating vertex. We also define a *total dominating set* S of a graph G to be a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to some vertex in S, and we write $\gamma_t(G)$ for the minimum size of such a set.

The Kneser graph $K(n, k)$ has as vertices the k-sets of $[n]$. Two vertices of $K(n, k)$ are adjacent if the k-sets are disjoint. Values of graph invariants for the Kneser graph are of interest to graph theorists. In a classic paper, Lovász [19] proved that the chromatic number of $K(n,k)$ is $k+2$, proving a long standing conjecture of Kneser [15].

In Chapter 8 we prove that $\gamma(K(n,k)) = \gamma_t(K(n,k)) = k+t+1$ when k^2+k-t, when $k^2+k-t\lfloor\frac{k}{2}\rfloor \le n < k^2+k-(t-1)\lfloor\frac{k}{2}\rceil$ and $t \le \lceil\frac{k}{2}\rceil$. We extend our construction of total dominating sets to smaller n, but in that range optimality is unknown.

Our work in this area also has applications to a classic problem in design theory. $M(n,k,l)$ is the minimum size of a family of k-element subsets of $[n]$ such that every l-element subset is contained in at least one of the k-sets. Since a set is adjacent in the Kneser graph $K(n,k)$ to thes sets contained in its complement, the total domination number of the Kneser graph is exactly $M(n,n-k,k)$. In 1963, Erdos and Hanani [9] conjectured that given k and l the value of $M(n,k,l)$ approaches the trivial lower bound $\binom{n}{l}/\binom{k}{l}$ as n approaches infinity. It was not until 1985 that Rödl [26] proved this conjecture, using the celebrated Rödl Nibble method. Our result gives exact values of these covering parameters for values of n that are bounded in terms of k.

1.3. Additional notation and terminology

In this section we review basic terminology and notation of graph theory used throughout the thesis. We let $V(G)$ and $E(G)$ denote the vertices and edges of a graph G, respectively. An edge is a vertex pair; we write uv for the edge with vertices u and v. The vertices of an edge are its *endpoints*. An edge *joins* its endpoints. A *simple* graph has at most one copy of each vertex pair as an edge, while in a *multigraph* we allow multiple edges between vertices. The *size* $e(G)$ of a graph is $|E(G)|$. The *order* $n(G)$ of a graph is $|V(G)|$. The *complement* of a simple graph G, written $\overline{G}$, is the simple graph with the same vertex set as G, such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

An *isomorphism* f between two graphs G and H is a function $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ if and only if $uv \in G$. We say two graphs are *isomorphic* if there is an isomorphism between them. We often use the same notation for a graph and its isomorphism class. We write nG to denote the graph consisting of n disjoint copies of G.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* of G is a subgraph H such that every edge of G with both endpoints in $V(H)$ is in $E(H)$. When $S \subseteq V(G)$ we write $G - S$ to denote the subgraph induced by the vertices $V(G) - S$. A *complete graph* or *clique* is a graph in which every pair of vertices forms an edge. We write K_n for the clique on n vertices. An *independent set* in a graph G is a subset $S \subseteq V(G)$ such that S does not contain both endpoints of any edge in G, that is, the subgraph induced by S is the complement of a clique.

A graph is *bipartite* if its vertex set can be partitioned into two independent sets, called *partite sets*. A *complete bipartite graph* has as edges all pairs consisting of one vertex from each partite set. A *complete multipartite graph* has a partition into partite sets such that every pair of vertices from different partite sets forms an edge. A complete multipartite graph is *equipartite* if each partite set has the same number of vertices.

When two vertices x and y form an edge, we say that they are *adjacent* or that they are *neighbors* and write $x \leftrightarrow y$. We write $N(x)$ for the *neighborhood* (set of neighbors) of a vertex x, and we define $N(X) = \left(\bigcup_{x \in X} N(x)\right)$ for the neighborhood of a set of vertices X. The *degree* of a vertex v, written $d(v)$, is the number of neighbors of v. The *maximum degree* of a graph G is the maximum of the degrees of vertices in G, written $\Delta(G)$. A graph is *even* if every vertex has even degree. If every vertex of a graph G has degree d, then G

is *d-regular.*

The *cartesian product* of two graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and an edge joining (g_1, h_1) and (g_2, h_2) if and only if 1) $g_1 = g_2$ and $h_1 \leftrightarrow h_2$ or 2) $g_1 \leftrightarrow g_2$ and $h_1 = h_2$. The *join* of two graphs G and H is the graph $G \vee H$ with vertex set $V(G) \cup V(H)$ and edge set consisting of the edges of G, the edges of H, and all pairs consisting of a vertex of G and a vertex of H.

A *walk* of length k is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for all i. The *endpoints* of a walk are its first and last vertices. A *trail* is a walk with no repeated edge. An *Eulerian* trail in a graph G is a trail that uses every edge of G. A *path* is a walk with no repeated vertex. A walk or trail is *closed* if its first vertex is the same as its last.

A *path* is a graph whose vertices can be listed in order so that the edges are precisely the pairs of consecutive vertices. The first and last vertices of a path are known as its *endpoints*; a path with endpoints x and y is an x, y-path. A *cycle* is a graph whose vertices can be listed cyclically so that the edges are precisely the pairs of consecutive vertices. We can specify a path in a simple graph by listing its vertices from either end. We can specify a cycle by listing its vertices cyclically, starting with any vertex and proceeding in either direction. A *spanning cycle* of a graph G is a cycle using every vertex in $V(G)$. Two paths are *internally-disjoint* if the only vertices they have in common are endpoints. We use P_n to denote the [isomorphism class of] paths with n vertices, similarly C_n denotes cycles with n vertices. A *chord* of a cycle C (resp. path P) in a graph G is an edge in G that has as endpoints two non-consecutive vertices of C (resp. P).

The *distance* $d(x, y)$ between two vertices x and y is the length of the shortest path between them. The *diameter* of a graph is the maximum of $d(x, y)$ over all pairs of vertices x, y. The *radius* of a graph G is $\min_{x \in V(G)} \max_{y \in V(G)} \{d(x, y)\}$. Any vertex x that minimizes $\max_{y \in V(G)} \{d(x, y)\}$ is a *center* of G.

Edges with a common vertex are *incident*. A *matching* is a set of edges in which no two are incident. A *perfect matching* of a graph G is a matching M in which every vertex in $V(G)$ is in some edge in M. A *decomposition* of a graph G is a partition of its edge set $E(G)$. Thus a proper edge-coloring of a graph is a decomposition of it into matchings.

A graph is *planar* if it can be drawn in the plane with no crossing edges. Such a drawing is called an *embedding*. A graph is *outerplanar* if it has an embedding in the plane such that there exists a face containing every vertex.

We denote the largest integer no larger than x by $\lfloor x \rfloor$; similarly $\lceil x \rceil$ denotes the smallest integer no smaller than x. We write $[n]$ for the set $\{1, 2, \ldots, n\}$. A *k-set* is a set with k elements. A *k-subset* of S is a k-set contained in S.

A graph G is connected if for all $x, y \in V(G)$ there is an x, y-path. A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ is not connected. A vertex cut of size 1 is a *cut-vertex*. A graph G is k-connected if every vertex cut of G has at least k vertices. A *block* of a graph G is a maximal connected subgraph of G that has no cut vertex.

In chapter 3 we will need Menger's theorem, which states that for any pair x, y of vertices in a k-connected graph there exist k pairwise internally-disjoint x, y-paths. We will also use the fact that every planar triangulation with at least 4 vertices is 3-connected.

In chapter 6 we will use that G has a closed Eulerian trail if and only if G is connected and even, and that G has a non-closed Eulerian trail if and only if G is connected and has exactly two vertices with odd degree.

In chapter 8 we will need Shannon's theorem [27], which states that a multigraph with maximum degree k is $\lfloor \frac{3k}{2} \rfloor$-edge-colorable.

A function $f(n)$ is in $\Theta(g(n))$ if there exist constants c_1 and c_2 such that for sufficiently large n, $c_1 g(n) \leq f(n) \leq c_2 g(n)$.

2. Path colorings of cartesian products of cycles

The *path-chromatic number* $\chi_P(G)$ of a graph G is the least number of colors with which the vertices of G can be colored so that the subgraph induced by each color class induces a disjoint union of paths. ("Disjoint union of paths" means "maximum degree 2 and no cycles.") We say that G is *path i-colorable* if $\chi_P(G) \le i$ and that G is *path i-chromatic* if $\chi_P(G) = i$. We define χ_{P_k} to be the minimum j such that G can be colored with j colors so that each color class is a disjoint union of paths with at most k vertices.

In this chapter we determine $\chi_P(G)$ when is G a cartesian product of cycles. Since $\chi(G) \le 3$ for such graphs, we always have $\chi_P(G) \le 3$, and the problem is to determine when $\chi_P(G) = 2$.

Lemma 2.1. *If G is a graph with $\chi_P(G) \ge 2$ and H is a bipartite graph, then $\chi_P(G\Box H) = \chi_P(G)$.*

Proof: Let f be a path $\chi_P(G)$-coloring of G, and let X and Y be the partite sets of H. Define a coloring g of $G \Box H$ by

$$g(v,w) = \begin{cases} f(v) & \text{if } w \in X \\ f(v)+1 \mod \chi_P(G) & \text{if } w \in Y. \end{cases}$$

Under g no two adjacent vertices with the same first coordinate have the same color, since two such vertices must have second coordinate in opposite partite sets of H. Vertices with the same second coordinate induce copies of G colored by f, which is a path coloring. Thus g is a path coloring of $G \Box H$. □

Lemma 2.1 implies that when G is a cartesian product of cycles, $\chi_P(G)$ depends only on the odd factors of G. In the following we thus assume (without loss of generality) that G is a cartesian product of odd cycles.

Weaver and West [31] observed that cartesian products of one or two odd cycles are path 2-colorable. For products of two odd cycles, they also showed that $\chi_{P_k}(G) = 2$ unless at least $k-2$ of the two odd factors have length 3. They also observed (see Lemma 2.1.1 that P_k 2-coloring becomes easier for products of odd cycles as the odd cycle lenths increase. They thus asked the following for products of three or more odd cycles:

(1) For a given number of factors, what is the largest value of k such that $\chi_{P_k}(G) = 3$ regardless of the length of the factors. Is this value finite?

(2) Let G be a cartesian product of $r \geq 3$ odd cycles and let t be the length of the shortest factor. Is there a value $t_1(r)$ such that $t \geq t_1(r)$ implies $\chi_{P_k}(G) = 2$ when k is sufficiently large?

We prove that no product of four or more odd cycles is path 2-colorable, that is, that the value desired in (1) is infinite when $r \geq 4$. We also give an affirmative answer to question (2) for products of three odd cycles.

2.1. Products of three or fewer odd cycles

We first prove that if a cartesian product of odd cycles is path 2-colorable, then so is the cartesian product of longer odd cycles. We denote the vertices of $C_{i_1} \square C_{i_2} \square \cdots \square C_{i_n}$ as vectors $(a_1, a_2, \ldots, a_n)$, where $1 \leq a_j \leq i_j$. The *jth level* of such a graph is the graph (isomorphic to $C_{i_1} \square C_{i_2} \square \cdots \square C_{i_n - 1}$) induced by vertices $(a_1, a_2, \ldots, a_{n-1}, j)$. A *slice* of such a graph is formed in a similar manner by fixing any coordinate.

Lemma 2.1.1. (Weaver-West [31]) *If* $\chi_{P_j}(C_{i_1} \square C_{i_2} \square \cdots \square C_{i_n}) = 2$*, then* $\chi_{P_j}(C_{i_1} \square C_{i_2} \square \cdots \square C_{i_{n-1}} \square C_{i_n + 2}) = 2$.

Proof: Let $G = C_{i_1} \square C_{i_2} \square \cdots \square C_{i_n}$ and $H = C_{i_1} \square C_{i_2} \square \cdots \square C_{i_n+2}$. Let $f : V(G) \rightarrow \{0, 1\}$

Figure 2.1.

be a path 2-coloring of G. Define a coloring $g : V(H) \to \{0, 1\}$ of H by

$$g(a_1, a_2, \ldots, a_n) = \begin{cases} f(a_1, a_2, \ldots, a_n), & \text{if } a_n \leq i_n; \\ 1 - f(a_1, a_2, \ldots, i_n), & \text{if } a_n = i_n + 1; \\ f(a_1, a_2, \ldots, i_n), & \text{if } a_n = i_n + 2. \end{cases}$$

Essentially, this duplicates level i_n of G, with the two copies separated by a third copy with the colors inverted. This operation does not introduce any new edges into the graphs induced by each color, while it actually breaks any monochromatic path or cycle that passes *through* level i, i.e. uses levels i_{n-1}, i_n and 1. □

Weaver and West [31] observed that Lemma 2.1, Lemma 2.1.1, and the colorings of Figure 2.1 imply that every cycle-product with at most two odd cycles is path 2-colorable. In the figures we show edges that "wrap around" as half edges, they actually connect vertices on opposite sides of the diagram.

We next show that the product of three odd cycles is path 2-colorable when each cycle has length at least 15. In fact, our colorings induce no monochromatic path with more than 5 vertices. This answers the second question of Weaver and West in the affirmative.

Theorem 2.1.2. *Let $G = C_a \square C_b \square C_c$ be the cartesian product of three odd cycles. If a, b, and c are each at least 15, then $\chi_P(G) = \chi_{P_5}(G) = 2$.*

Proof: Begin with the coloring h of $H = C_5 \square C_5$ shown in Figure 2.2. Use it to define a coloring g of $G = C_5 \square C_5 \square C_5$ by $g(a_1, a_2, a_3) = h(a_1 - a_3, a_2 - a_3)$. Thus, each level of

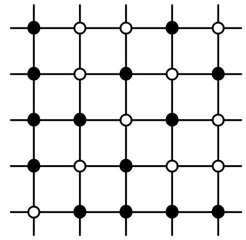

Figure 2.2.

G is colored as in Figure 2.2, but the colorings are shifted diagonally from level to level.

Now apply the operation of Lemma 2.1.1 to each slice of G in each dimension to obtain a coloring g' of $G' = C_{15} \square C_{15} \square C_{15}$. As noted in the proof of Lemma 2.1.1, there are no monochromatic paths or cycles in G' that pass *through* any slice in any direction. Thus, every monochromatic path or cycle in this coloring must appear in a 2 by 2 by 2 cubelet that also occurs as a 2-colored 2 by 2 by 2 cubelet of G.

By inspection of the coloring g, we can see that for each vertex in G there is at least one direction in which both of its neighbors are of the opposite color. For the vertices where this does not occur within a level, the nearest vertices along increasing diagonals have the opposite color. The diagonal shift from level to level causes the property to hold in the third direction.

Thus, in every 2 by 2 by 2 cubelet of G (and hence in G'), each vertex has at most two neighbors with its own color. Among such colorings of the 2 by 2 by 2 cube, only those of Figure 2.3 (modulo the switching of colors) have monochromatic cycles. (In Figure 2.3 the first coordinate is horizontal, the second extends diagonally, and the third is vertical.) For each such colored cubelet, we note the coloring of a subgraph of H that would cause that cubelet to appear in G. Because the coloring of H is shifted diagonally from level to level when coloring G, the front left top vertex of each cubelet of G has the same color as the

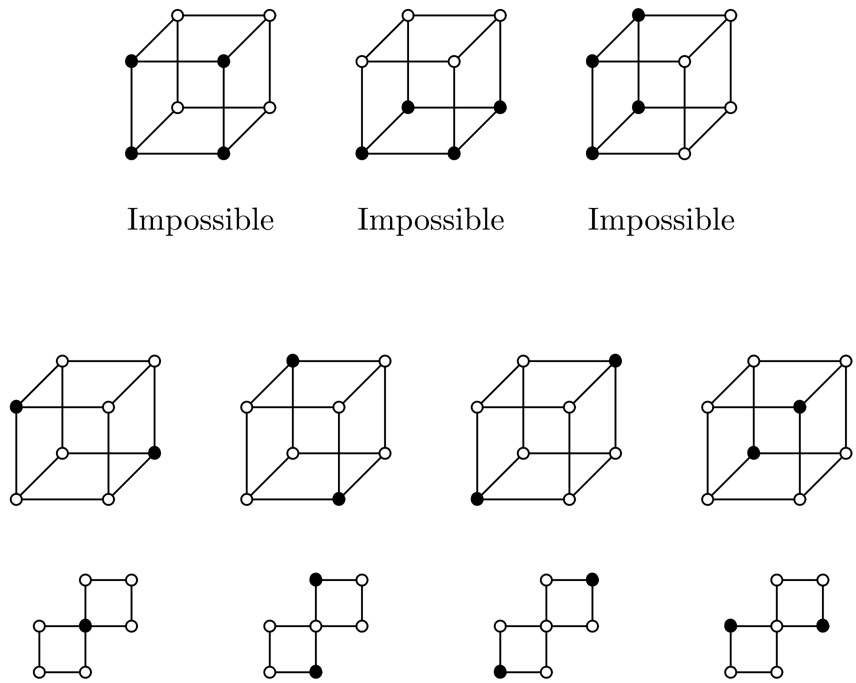

Figure 2.3.

back right bottom vertex. Thus, the cubelets in the top row of Figure 2.3 cannot occur in G. Since h contains none of the colorings corresponding to cubelets in the bottom row of Figure 2.3 (or the colorings obtained by switching colors), we are assured that G' has no monochromatic cycles.

Additionally, since the longest monochromatic path in any path coloring of a 2 by 2 by 2 cube with 2 colors has length 5, (and, as noted, each monochromatic path in G' occurs in such a cubelet), we have proved the stronger result that $\chi_{P_5}(G') = 2$. Lemma 2.1.1 now implies that $\chi_{P_5}(C_a \square C_b \square C_c) = 2$ whenever $a \geq 15$, $b \geq 15$, and $c \geq 15$. $\square$

Theorem 2.1.3. *If G is a cartesian product of three odd cycles, then $\chi_{P_4}(G) = 3$.*

Proof: If every 2 by 2 by 2 cubelet has four vertices of each color, then each color is used on the same number of vertices, because each vertex appears in the same number of

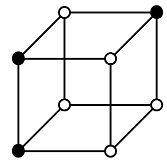

Figure 2.4.

cubelets. This can not happen since the total number of vertices is odd. Thus, every path 2-coloring of $C_a \square C_b \square C_c$ has a cubelet that does not have four vertices of each color. The only path 2-coloring for such a cubelet (up to symmetry and switching of colors) is shown in Figure 2.4, and this coloring induces a monochromatic path of length 5. □

Theorem 2.1.4. *If $G = C_3 \square C_b \square C_c$, where b and c are odd, then $\chi_P(G) = 3$.*

Proof: Consider a hypothetical path 2-coloring of $C_3 \square C_b \square C_c$. Color $C_b \square C_c$ by projection, giving (i, j) the color used on the majority of $(1, i, j)$, $(2, i, j)$, and $(3, i, j)$. Because $C_b \square C_c$ has an odd number of vertices, it cannot happen that each 2 by 2 square has 2 of each color, using symmetry as in the proof of Theorem 2.1.3. Thus, we may assume that $(1, 1)$, $(2, 1)$, and $(1, 2)$ have the same color.

Let H be the subgraph of $C_3 \square C_b \square C_c$ induced by vertices with second and third coordinate equal to 1 or 2. By listing as successive columns the vertices whose last two coordinates are (2,1), then (1,1), then (1,2), then (2,2), we see that H is isomorphic to $C_4 \square C_3$, colored so that the first three columns H each have a majority of the same color (shown in black in Figure 2.5). Without loss of generality, we may assume that the first two columns are colored as in the first diagram of Figure 2.5. Since each vertex has no more than two neighbors of its own color, the middle row vertices in the other columns must have the other color (white), as shown in the second diagram. Since the majority

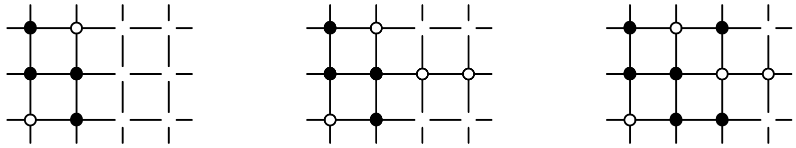

Figure 2.5.

color in the third column is the same as in the first two columns, the coloring must be as in the third diagram of Figure 2.5. Avoiding degree 3 and cycles in black forces the two remaining vertices to be white, but then the subgraph induced by the white vertices has a vertex of degree 3. Thus our hypothetical path 2-coloring of $C_3 \square C_b \square C_c$ cannot exist. $\square$

When attempting to path 2-color products of three odd cycles, longer cycles give us more freedom, but limiting the size of induced monochromatic paths gives us less freedom. We have shown that if the smallest cycle has length at least 15 we can always path 2-color the product without inducing monochromatic paths with 6 vertices. We have also shown that if the smallest cycle has length 3, then the product is not path 2-colorable. Which of the remaining products of three odd cycles are path 2-colorable? What is the minimum k for which we can achieve $\chi_{P_k}(G) \leq 2$? Computer analysis (exhaustive search) has shown the following:

$\chi_P(C_5 \square C_5 \square C_c) = 3$ for odd $c \leq 9$.

$\chi_{P_{13}}(C_5 \square C_5 \square C_{15}) = 3$ but $\chi_{P_{14}}(C_5 \square C_5 \square C_{11}) = 2$

$\chi_{P_8}(C_5 \square C_7 \square C_7) = 3$ but $\chi_{P_9}(C_5 \square C_7 \square C_7) = 2$.

Thus, the minimal triples (a, b, c) such that $C_a \square C_b \square C_c$ is path 2-colorable are $(5, 7, 7)$ and $(5, 5, 11)$. These triples are minimal in the sense that all larger products of three odd cycles are path 2-colorable, by Lemma 2.1.1.

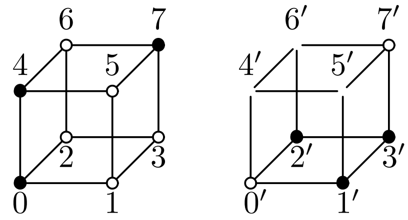

Figure 2.6.

2.2. Products of four or more odd cycles

Theorem 2.2.1. *If G is a cartesian product of four or more odd cycles, then $\chi_P(G) = 3$.*

Proof: Let $G = C_a \,\square\, C_b \,\square\, C_c \,\square\, C_d$, and consider a hypothetical path 2-coloring g of G. For $1 \leq l \leq d$, define $H_l = C_a \,\square\, C_b \,\square\, C_c$, and let $h_l(i, j, k) = g(i, j, k, l)$. For each l, h_l is a proper coloring of H_l. As noted in the proof of Theorem 2.1.2, every path 2-coloring of H_l must contain a 2 by 2 by 2 cubelet colored as in Figure 2.4. Find such a cubelet in H_1. Figure 2.6 depicts such a cubelet on the left and the corresponding cubelet of H_2 on the right (so $0 \leftrightarrow 0', 1 \leftrightarrow 1'$, etc.).

Vertices $1'$, $2'$, $3'$ must be black, otherwise one of 1,2,3 will have three neighbors of its own color. Then $0'$ must be white to avoid the black cycle $(0', 1', 2', 3')$. Also, $7'$ must be white so that $3'$ has only two black neighbors. Now any way of coloring the three remaining vertices $4', 5', 6'$ must use exactly two black and one white, otherwise there is a white cycle or a white vertex with three white neighbors.

Each such coloring of the cubelet of H_2 looks like the coloring of the cubelet of H_1 with the colors inverted to have five black and three white vertices, although it may have a different orientation. Similarly, the corresponding cubelet of H_3 again has a similar coloring, with the *same* majority color as the cubelet of H_1. Repeating the analysis shows that every copy of this cubelet has an unbalanced coloring, with the majority color alternating as we traverse the cycle of copies. However, we cannot alternate two options along an odd cycle. $\square$

3. Path colorings of toroidal graphs

In this chapter we show that every toroidal graph can be 3-colored so that two of the color classes induce disjoint unions of paths and the third induces a disjoint union of paths and at most one cycle. We will need the following definitions.

A *plane graph* is an embedding of a planar graph in the plane, and a *torus graph* is an embedding of a toroidal graph in the torus. If every face of a plane graph G except one is a triangle, then we say that G is *weakly triangulated.* Usually in such an embedding the non-triangular face appears as the unbounded region, called the *outer face.* Vertices not on the outer face of a plane graph are *interior* vertices. A plane graph G is *outerplanar* if all vertices of G lie on a single face. We often specify a path or cycle by listing its vertices so that consecutive vertices are adjacent. In simple graphs, this introduces no ambiguity. If P and P' are paths that have no vertices in common and the last vertex of path P is adjacent to the first vertex of P' we write $P{\cdot}P'$ to denote the path consisting of P and P' and the edge from the end of P to the beginning of P'. If also the first vertex of P is adjacent to the last vertex of P', then $P{\cdot}P'$ denotes the cycle consisting of P and P' and the two intervening edges.

3.1. Path coloring of planar graphs

We first need to develop a lemma. Independently discovered, it is equivalent to one used by Poh [24] in proving that $\chi_P(G) \leq 3$ for each planar graph G). The second condition of the lemma is a technical loading of the induction hypothesis that simplifies the inductive proof.

Lemma 3.1.1. *Let G be a weakly triangulated plane graph whose outer face is a cycle*

$C = (x_1, x_2, \ldots, x_p)$. *If* C *is* 2*-colored so that each color class induces a non-empty path, then the coloring can be extended to a* 3*-coloring of* G *such that*

1. *each color class induces a disjoint union of paths, and*
2. *the (precolored) vertices of* C *are adjacent to no additional (interior) vertices of their own color.*

Proof: We use induction on the number of vertices. For graphs with at most three vertices the statement is trivial.

For larger graphs, consider a graph and a coloring of the outer face C as specified in the hypotheses. We may assume that C has length p, with $x_1, x_2, \ldots, x_q$ blue and $x_{q+1}, x_{q+2}, \ldots, x_p$ red).

Suppose first that there is an edge other than x_1x_p or x_qx_{q+1} joining a blue vertex and a red vertex, say x_bx_r, where $1 \leq b \leq q$ and $q+1 \leq r \leq p$. In this case, apply the induction hypothesis to the subgraphs obtained from the embeddeing by extracting the cycles $(x_r, x_{r+1}, \ldots, x_p, x_1, x_2, \ldots, x_b)$ and $(x_b, x_{b+1}, \ldots, x_q, x_{q+1}, \ldots, x_{r-1}, x_r)$ with their interiors (see Figure 3.1). Since these colorings agree on the vertices common to both, we can combine them to form a coloring of G. Condition 2 guarantees that the two pieces of each path have not been extended into the interior. Thus, they still fit together to form a single path when we combine the colorings, so Condition 1 is satisfied. Condition 2 holds for the coloring of G since it holds for the colorings of the subgraphs.

If there is no such edge x_bx_r, let y be the third vertex on the interior face bounded by x_1x_p, and let z be the third vertex on the face bounded by x_1x_{q+1}.

Let $P = (y, y_1, \ldots, y_l, z)$ be a chordless y,z-path in $G - V(C)$. Such a path exists because G is weakly triangulated and has no edge from a blue vertex to a red vertex. If

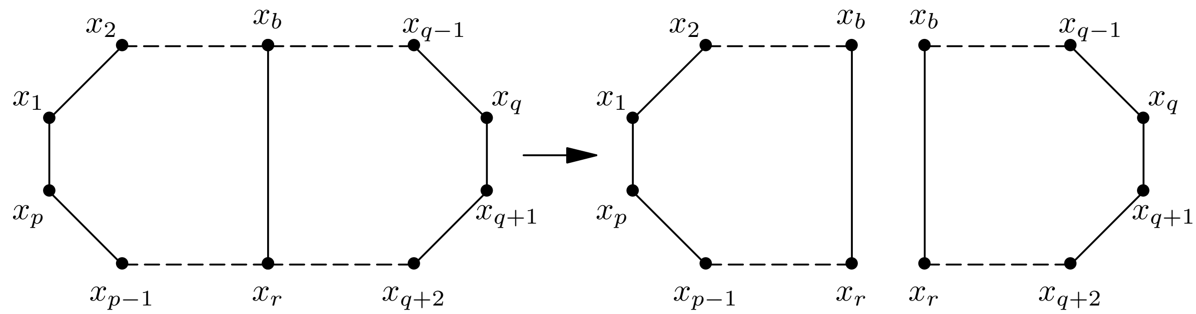

Figure 3.1

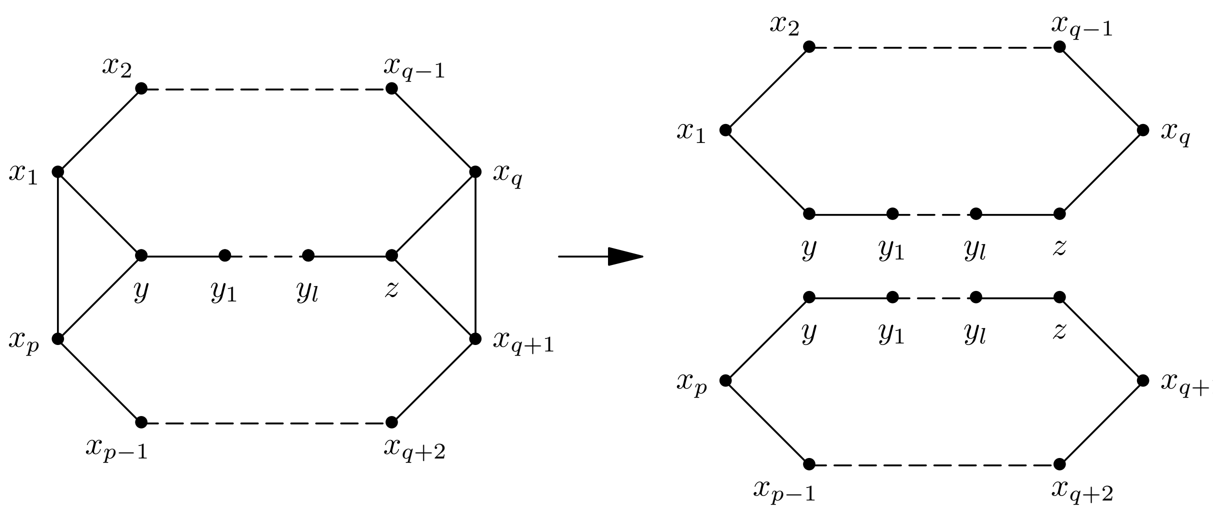

Figure 3.2.

$y = z$ then $P = (y)$. Now we can color the vertices of P green and apply the inductive hypothesis to the subgraphs of G obtained from the embedding by extracting the cycles $(y, y_1, \ldots, y_l, z, x_q, x_{q-1}, \ldots, x_2, x_1)$ and $(y, y_1, \ldots, y_l, z, x_{q+1}, x_{q+2}, \ldots, x_p)$ with their interiors (see Figure 3.2).

Again, these colorings agree on common vertices and thus can be combined to form a coloring of G. Condition 2 guarantees that P was not extended in either coloring, so this coloring of G is a path coloring, and Condition 1 is satisfied. Condition 2 is also satisfied, since it holds for the colorings of the subgraphs. □

First, we note that this lemma immediately yields the theorem of Poh and Goddard. We subsequently use the lemma in the next section.

Theorem 3.1.2. (Poh [24], Goddard [11]) *If G is a planar graph, then $\chi_P(G) \leq 3$.*

Proof: Since adding edges does not make G easier to color, we may add edges to an embedding of G to make it a triangulated graph, color two vertices of the outer face blue and the third red, and apply Lemma 3.1.1. □

3.2. Coloring torus graphs

In topology, we can get a sphere from a torus by cutting the torus and filling in the two holes with discs. In the following theorem we perform basically the same operation to a graph, in order to get a planar graph from a toroidal graph. We cut along a non-contractible cycle, and the resulting embedded graph on the sphere has two copies of this cycle, each bounding a face. We will subsequently contract the copies of this cycle and apply Lemma 3.1.1 to pieces of the resulting plane graph.

Theorem 3.2.1. *If G is a torus graph with a chordless noncontractible cycle C, then G can be 3-colored so that two color classes each induce a disjoint union of paths, while the third color class induces a disjoint union of paths and the cycle C*

Proof: Let G be a torus graph, and C be a chordless noncontractible cycle of G. We construct H, a planar graph, by cutting the embedding of G along C. This replaces C with two copies of itself, C_1 and C_2. Each edge from a vertex x not in C to a vertex y in C is replaced with an edge from x to one of the copies of y . "Cutting" a cycle in this manner turns a toroidal graph into a planar graph where the regions enclosed by C_1 and

C_2 are faces. Now form H' by contracting C_1 and C_2 to vertices $\hat{x}$ and $\hat{y}$ and then adding any edges (other than $\hat{x}\hat{y}$) necessary to triangulate the resulting planar graph. It is now sufficient to 3-color H' so that each of the color classes induces a disjoint union of paths and $\hat{x}$ and $\hat{y}$ are the same color but are not adjacent to any other vertex of that color. After transferring this coloring back to G by giving all of C that color, C will be the only cycle induced by a color class. Note that $\hat{x}$ and $\hat{y}$ are not adjacent, since C is chordless in G.

Since H' is a triangulation, the neighbors of any vertex form a cycle in H'. Given a_1 and a_2 adjacent to a, we will use the interval notation $[a_1, a_2]_a$ (resp. $(a_1, a_2)_a$) where a_1 and a_2 are adjacent to some vertex a to denote the subset of neighbors of a from a_1 to a_2 in clockwise order, including (resp. excluding) a_1 and a_2. We will similarly write $P[p_j, p_k]$ where p_j and p_k are vertices on some path P to denote the subpath of P from p_j to p_k, including p_j and p_k.

We now have two cases:

Case 1. $|N(\hat{x}) \cap N(\hat{y})| \geq 2$, where $z_1, z_2, \ldots, z_n$ are the common neighbors of $\hat{x}$ and $\hat{y}$ listed in clockwise order around $\hat{x}$ (which is necessarily counterclockwise order around $\hat{y}$). Although $z_1, z_2, \ldots, z_n$ need not induce a cycle, we will treat indices modulo n.

For each i with $1 \leq i \leq n$, we must color the vertices in the region bounded by the cycle ($\hat{x}$, z_i, $\hat{y}$, z_{i+1}). To do so, we find chordless paths Y_i from z_{i+1} to z_i in $[z_{i+1}, z_i]_{\hat{y}}$ and X_i from z_i to z_{i+1} in $[z_i, z_{i+1}]_{\hat{x}}$. Then, we color $V(Y_i) - \{z_{i+1}, z_i\}$ red, the interior vertices of $V(X_i) - \{z_i, z_{i+1}\}$ blue, and $\hat{x}$ and $\hat{y}$ green. The vertices in $\{z_i : 1 \leq i \leq n\}$ may be colored blue or red arbitrarily, as long as both red and blue are used at least once so that this set does not form a monochromatic cycle with the paths $\{X_i\}$ or $\{Y_i\}$. Now,

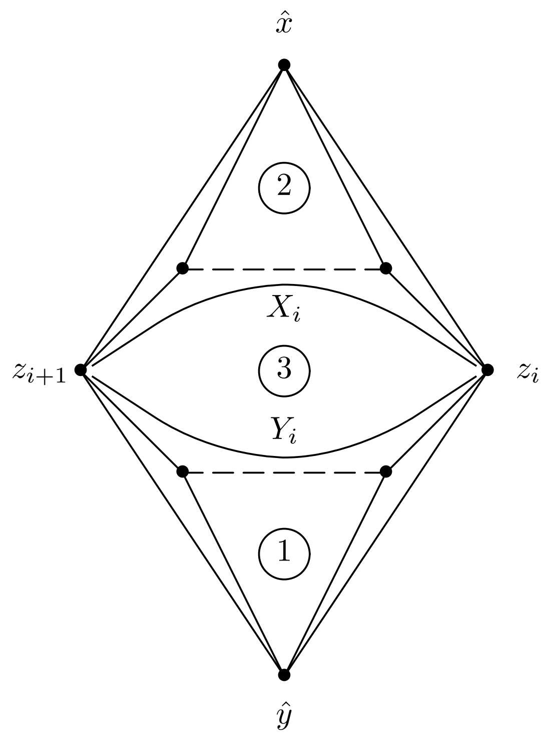

Figure 3.3.

each color class induces a disjoint union of paths, and we can complete the coloring of H' by applying Lemma 3.1.1 to the following for each i with $1 \leq i \leq n$ (see Figure 3.3.):

1. The region bounded by $\hat{y}$ and the interior vertices of Y_i.
2. The region bounded by $\hat{x}$ and the interior vertices of X_i.
3. The region bounded by X_i and Y_i.

The combine to form a coloring of H', since they agree on common vertices. Note that Condition 2 of Lemma 3.1.1 is needed here. We must ensure that none of the precolored paths aquires neighbors in its color off these paths, causing a vertex of degree three. Condition 1 of Lemma 3.1.1 then guarantees that the resulting coloring is a path coloring.

Case 2. $|N(\hat{x}) \cap N(\hat{y})| \leq 1$. Since H' is triangulated, it is 3-connected, so we can apply Menger's theorem and find three pairwise internally-disjoint induced paths from $\hat{x}$ to $\hat{y}$. Let

$P = (p_1, p_2, \dots, p_m)$ and $Q = (q_1, q_2, \dots, q_n)$ be the paths formed by the internal vertices of two such $\hat{x}$,$\hat{y}$-paths, where P is chosen to be as short as possible. If $|N(\hat{x}) \cap N(\hat{y})| = 1$, then P is the common neighbor of $\hat{x}$ and $\hat{y}$.

At this point, we would like to have two paths that (with $\hat{x}$ and $\hat{y}$) partition the plane into regions where Lemma 3.1.1 applies if we use one color on each path. The obvious candidates are shown in Figure 3.4, where A_p and A_q are paths in $N(\hat{x})$ and B_p and B_q are paths in $N(\hat{y})$. Note that p_1, q_1, p_m, and q_n do not appear in the paths we call A_p, A_q, B_p, B_q. The regions bounded by the cycles $\hat{x}{\cdot}A_p$, $\hat{x}{\cdot}A_q$, $\hat{y}{\cdot}B_p$, $\hat{y}{\cdot}B_q$, $P{\cdot}B_p{\cdot}Q{\cdot}A_q$, and $P{\cdot}B_q{\cdot}Q{\cdot}A_p$ then bound all uncolored vertices in the graph. This approach does not work without further care. If there is an (say) edge from some vertex of P other than p_1 to some vertex of A_p then P and A_p do not form an induced path. Our solution is to use such a chord if it exists, skipping vertices bypassed by the chord. We then must extend A_q to use the vertices no longer used in A_p (see, for instance, Figure 3.6) Note that a chord from p_1 to some internal vertex of Q does not cause problems unless A_q is extended in this manner. The following four subcases of case 2 describe exactly how we must use such chords and how to extend A_p and A_q so that the pieces can be put together to form induced paths.

We define $P^+ = \{p_2, p_3, \dots, p_m\}$ and $Q^+ = \{q_2, q_3, \dots, q_n\}$. Define paths P', Q', A_p and A_q as follows:

Case 2a. *P^+ and $(p_1, q_1)_{\hat{x}}$ are pairwise non-adjacent and Q^+ and $(q_1, p_1)_{\hat{x}}$ are pairwise non-adjacent.* (This is the case where there are no "bad" chords.) Let $P' = P$, $A_p = (p_1, q_1)_{\hat{x}}$, $Q' = Q$, and $A_q = (q_1, p_1)_{\hat{x}}$ (Figure 3.5).

Case 2b. *Q^+ and $(q_1, p_1]_{\hat{x}}$ are pairwise non-adjacent, but at least one edge joins P^+ and*

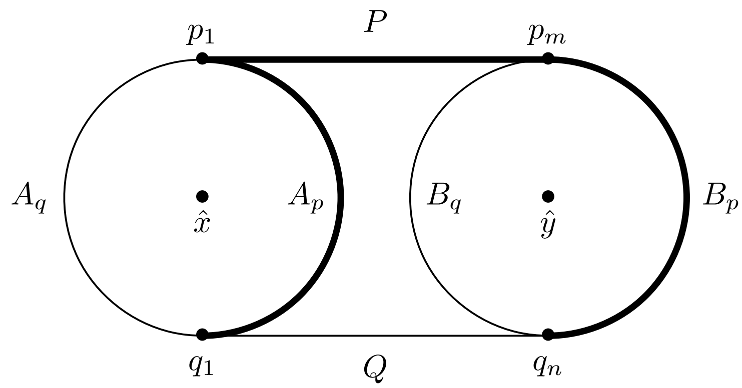

Figure 3.4

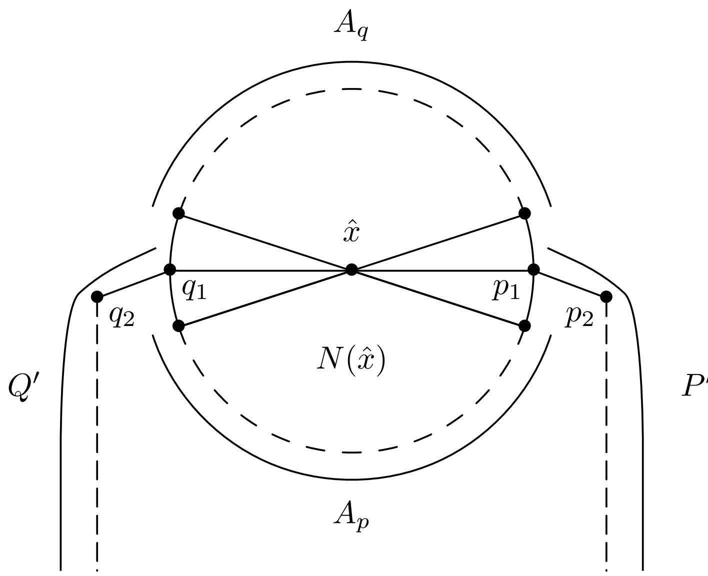

Figure 3.5 Case 2a.

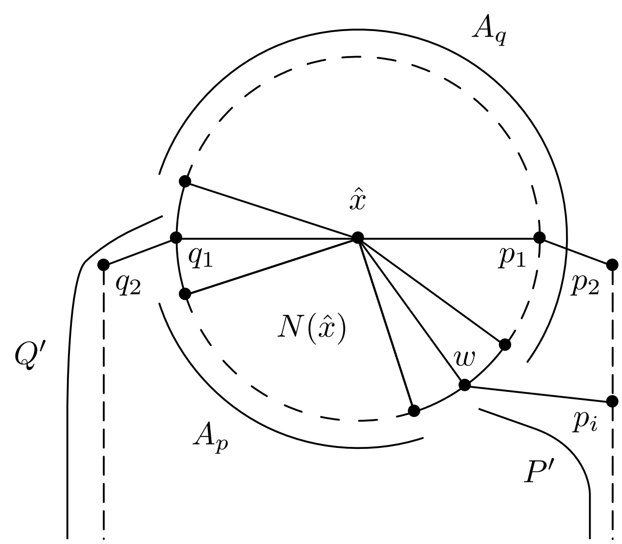

Figure 3.6: Case 2b.

$(p_1, q_1)_{\hat{x}}$. (This case forbids edges from p_1 to Q^+.) Let w be the last vertex (in clockwise order around $\hat{x}$) in $(p_1, q_1)_{\hat{x}}$ that has a neighborin P^+, and let $i = \max\{j : p_j \in N(w)\}$. Define $P' = w{\cdot}P[p_i, p_m]$, $A_p = (w, q_1)_{\hat{x}}$, $Q' = Q$, and $A_q = (q_1, w)_{\hat{x}}$. (Figure 3.6).

Case 2c. *P^+ and $(p_1, q_1]_{\hat{x}}$ are non-adjacent, but at least one edge joins Q^+ and $(q_1, p_1)_{\hat{x}}$.* (This case forbids chords from Q_1 to P^+ and is symmetric to Case 2b.) Let w' be the last vertex (in clockwise order around $\hat{x}$) in $(q_1, p_1)_{\hat{x}}$ that has a neighbor in Q^+, and let

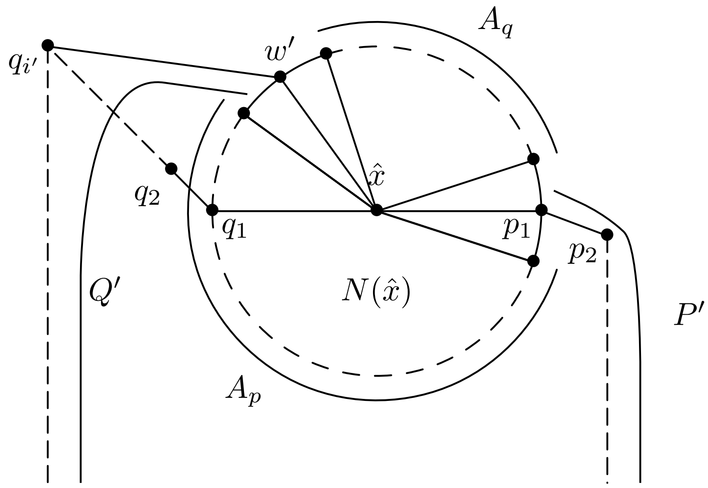

Figure 3.7: Case 2c.

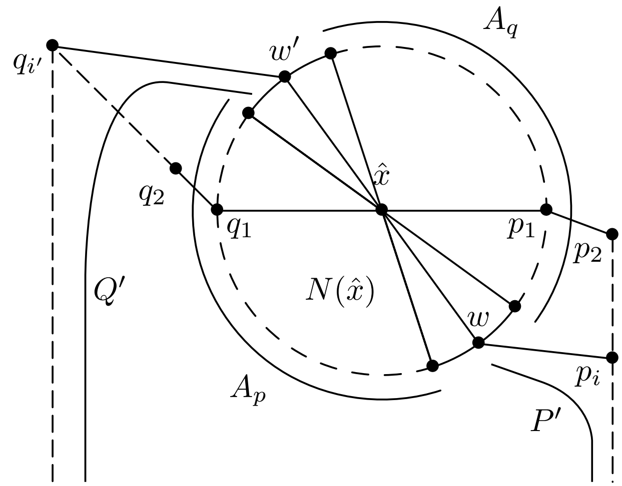

Figure 3.8: Case 2d.

$i' = \max\{j : q_j \in N(w')\}$. Define $P' = P$, $A_p = (p_1, w')_{\hat{x}}$, $Q' = w' \cdot Q[q_{i'}, q_n]$, and $A_q = (w', p_1)_{\hat{x}}$. (Figure 3.7).

Case 2d. *At least one edge joins P^+ and $(p_1, q_1]_{\hat{x}}$ and at least one edge joins Q^+ and $(q_1, p_1]_{\hat{x}}$.* (Either Case 2a or Case 2d may be used when all edges from P^+ to $(p_1, q_1]_{\hat{x}}$ involve q_1 and all edges from Q^+ to $(q_1, p_1]_{\hat{x}}$ involve p_1.) Let w be the last vertex (in clockwise order around $\hat{x}$) in $(p_1, q_1]_{\hat{x}}$ that has a neighbor in P^+, let $i = \max\{j : p_j \in N(w)$, let w' be the last vertex (in clockwise order around $\hat{x}$) in $(q_1, p_1]_{\hat{x}}$ that has a neighbor in Q^+, and let $i' = \max\{j : q_j \in N(w')\}$ Define $P' = w \cdot P[p_i, p_m]$, $A_p = (w, w')_{\hat{x}}$, $Q' = w' \cdot Q[q_{i'}, q_n]$, and $A_q = (w', w)_{\hat{x}}$. (Figure 3.8).

Now let A'_p be a chordless path with the same endpoints as A_p and $V(A'_p) \subseteq V(A_p)$. We obtain A'_p by using chords of A_p if they exist, skipping the vertices on A_p between the endpoints of chords. Similarly, obtain A'_q from A_q.

In a manner similar to the above, we start with P' and Q' instead of P and Q and use the neighbors of $\hat{y}$ to define chordless paths P'' and Q'' from P' and Q' and chordless

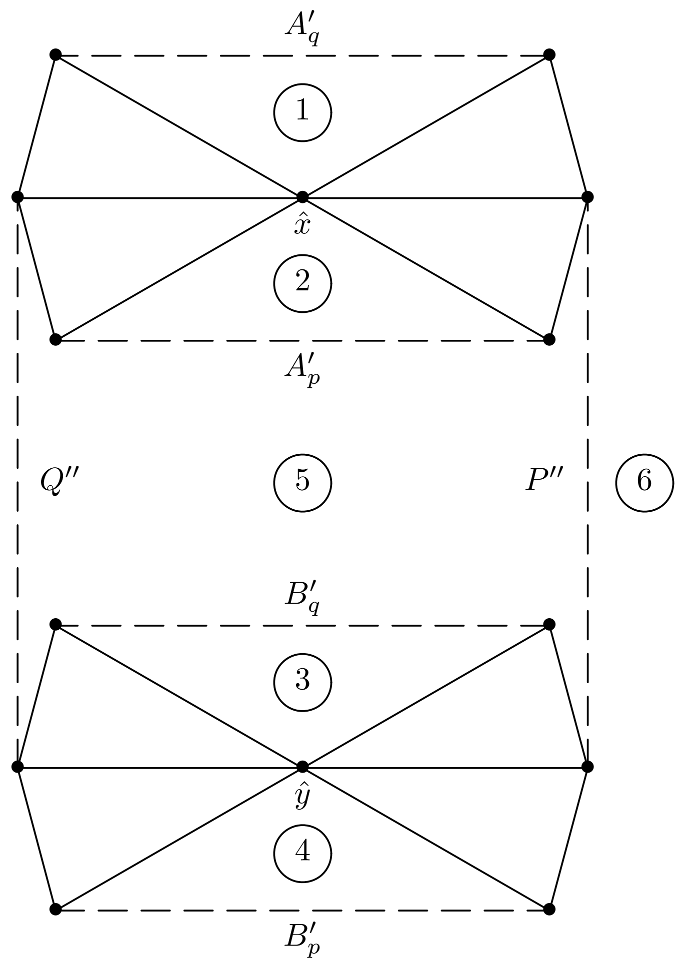

Figure 3.9.

paths B'_p and B'_q within $N(\hat{y})$. Note that A'_p and A'_q share no vertices with B'_p and B'_q, since if $N(\hat{x})$ and $N(\hat{y})$ intersect, they do so in only the vertex, $p_1 = P' = P''$.

The cycle $\hat{x} \cdot P'' \cdot \hat{y} \cdot Q''$ separates A'_p from B'_p and A'_q from B'_q (see Figure 3.9). This, together with the choices of A'_p, B'_p, A'_q, and B'_q, assures us that $A'_p \cdot P'' \cdot B'_p$ is a chordless path, as is $A'_q \cdot Q'' \cdot B'_q$. Color the vertices of $A'_p \cdot P'' \cdot B'_p$ red, the vertices of $A'_q \cdot Q'' \cdot B'_q$ blue, and $\hat{x}$ and $\hat{y}$ green. Now, regions bounded by the following six cycles contain all of the uncolored vertices of H'.

1. $\hat{x} \cdot A'_q$.

2. $\hat{x} \cdot A'_p$.

3. $\hat{y} \cdot B'_q$.

4. $\hat{y} \cdot B'_p$.

5. $P'' \cdot B'_p \cdot Q'' \cdot A'_q$.

6. $P'' \cdot B'_q \cdot Q'' \cdot A'_p$.

Since each of these cycles satisfies the hypotheses of Lemma 3.1.1, we can apply that lemma to color the vertices in each of these regions. Again, since these colorings agree on common vertices, we can combine them to get a coloring of H'. Condition 2 of Lemma 3.1.1 implies that no pre-colored path received a neighbor with the same color. Together with Condition 1, this implies that the resulting coloring of H' is a path coloring in which $\hat{x}$ and $\hat{y}$ have no green neighbors. As we observed at the start of the proof, this yields the desired coloring of G. □

3.3. Other Results

Lemma 3.1.1 can also be used to show that some nonplanar graphs have path chromatic number at most 3. The *crossing number* of a graph is the least k such that the graph can be drawn on the plane with k edge crossings. The following result can be used to show that graphs with crossing number at most 1 are path 3-colorable. It would be interesting to find the maximum t such that all graphs with crossing number at most t are path 3-colorable.

Corollary 3.3.1. *Let G be a simple graph. If there exists a set of edges S of G and a planar embedding of $G \setminus S$ that satisfies the two conditions below, then $\chi_P(G) \leq 3$.*

1. The endpoints of edges in S are all on the outer face.

2. The outer face can be 2-colored such that $G \setminus S$ satisfies the hypothesis of Lemma 3.1.1 *and each edge in S connects vertices of opposite color.*

Proof: Apply Lemma 3.1.1 to $G \setminus S$ with the given coloring. Since edges in S connect vertices of opposite color, the resulting coloring of G is valid. □

Corollary 3.3.2. *If G has crossing number 1, then $\chi_P(G) \leq 3$.*

Proof: Let S be the two crossing edges vw and xy in some drawing of G on the plane. Add to G all edges in $\{vx, vy, wx, wy\}$ that are not already present. This can be done without additional crossings, since vw and xy form the only crossing in G. Color v and x blue, and color w and y red. Redraw G (if necessary) so that the face bounded by v, w, x and y is the outer face. Corollary 3.3.1 now applies. □

Heawood [12] showed that $\chi(G) \leq 7$ when G is a toroidal graph, but more can be said: toroidal graphs can be 7-colored so that only one vertex has the seventh color. As mentioned by Dirac [8], P. Ungar proved that any toroidal graph requiring 7 colors contains K_7 as a subgraph. Theorem 3.2.1 yields an elementary proof of a weaker form of this statement:

Theorem 3.3.3. *Let G be a toroidal graph. If G has an embedding that has a chordless even noncontractible cycle, then $\chi(G) \leq 6$. Otherwise, G can be colored with seven colors so that the one color class consists of only one vertex.*

Proof: Since a disjoint union of paths is bipartite, we can properly color G by using two colors for each color class in the coloring from Theorem 3.2.1, unless the cycle in the third color class is an odd cycle. In that case, we need one vertex of another color to properly color that cycle. □

We believe that the condition "G has no chordless even noncontractible cycle" implies that G contains K_7 as a subgraph. This would give another proof of Ungar's Theorem.

4. Path list colorings of planar graphs

In this chapter we combine the path chromatic number (discussed in chapters 2 and 3) with the idea of *list coloring* to get the *list path chromatic number* $\hat{\chi}_P$. We show that planar graphs have list path chromatic number at most 3.

Recall that $\hat{\chi}_P(G) \leq k$ if, for every assignment of lists of k colors to $V(G)$, a color can be chosen for each vertex from its list so that each color class induces a disjoint union of paths.

Theorem 4.1. *If G is a planar graph, then $\hat{\chi}_P(G) \leq 3$.*

We actually prove a stronger statement as given in the following lemma. We show that we can find a more restrictive coloring, even when some lists have fewer than three colors. When f is a coloring of a graph, we define $d_f(x)$ to be the number of neighbors of x that receive the same color as x. A *cut-vertex* of a connected graph G is a vertex v such that $G - v$ has at least two components. A cut-vertex v *separating* two vertices x and y is a vertex v such that x and y are in different components of $G - v$.

Lemma 4.2. *Let G be a connected planar graph with distinguished vertices x and y (not necessarily distinct) on the outer face, and let S be the (possibly empty) set of cut-vertices separating x from y. If L is an assignment of lists to the vertices of G such that*

$$|L(v)| \geq \begin{cases} 1, & \text{if } v \in S \cup \{x, y\} \\ 2, & \text{if } v \text{ is any other vertex on the outer face} \\ 3, & \text{if } v \text{ is an interior vertex,} \end{cases}$$

then there is a path coloring f of G such that 1) $f(v) \in L(v)$ for all $v \in V(G)$, and 2) $d_f(x), d_f(y) \leq 1$.

Before we prove Lemma 4.2 we prove that its hypotheses hold in certain situations. We then use this in the inductive proof of Lemma 4.2 itself.

Proposition 4.3. *Let H be a plane graph with P a chordless path consisting of consecutive vertices on the outer face of H so that $H - V(P)$ is connected. Also let L be an assignment of lists to the vertices of G such that*

$$|L(v)| \geq \begin{cases} 1, & \text{if } v \in V(P) \\ 2, & \text{if } v \text{ is any other vertex on the outer face} \\ 3, & \text{if } v \text{ is an interior vertex.} \end{cases}$$

Suppose there is a common color c with $c \in L(v)$ for all $v \in V(P)$. Let $H' = H - V(P)$ and define

$$L'(v) = \begin{cases} L(v), & \text{if } v \text{ has no neighbors in } P \\ L(v) - c, & \text{if } v \text{ has at least one neighbor in } P \end{cases}$$

Then there exist x' and y' in H' so that H', L', x', and y' satisfy the hypothesis of Lemma 4.2.

Proof: Figure 4.1 depicts a typical example, where the color list $\{a, b, c\}$ is denoted *abc*. Let the vertices along the outer face of H be denoted $z_1, z_2, \ldots, z_l$ in clockwise order. (Note that a cut-vertex may appear repeatedly in the list of vertices along the outer face). We will treat subscripts modulo l. Without loss of generality, we may assume P begins at z_2 and proceeds clockwise. Define q so that z_q is the last vertex of P, and let $x' = z_1$ and $y' = z_{q+1}$

We claim that H', L', x', and y' satisfy the hypotheses of Lemma 4.2. Let S be the set of cut-vertices separating x' from y'. Since list sizes went down by at most 1 and every vertex not in P began with a list size of at least 2, all vertices v on the outer face of H' have $|L'(v)| \geq 1$. All interior vertices v of H' have $|L'(v)| \geq 3$ because if a vertex was adjacent to P it is on the outer face of H'.

Thus, we need only show that $|L'(v)| = 1$ requires $v \in S \cup \{x', y'\}$. Assume that there is a vertex $v \notin \{x', y'\}$ with $|L'(v)| = 1$. Since $|L'(v)| \geq |L(v)| - 1$, we must have $|L(v)| = 2$.

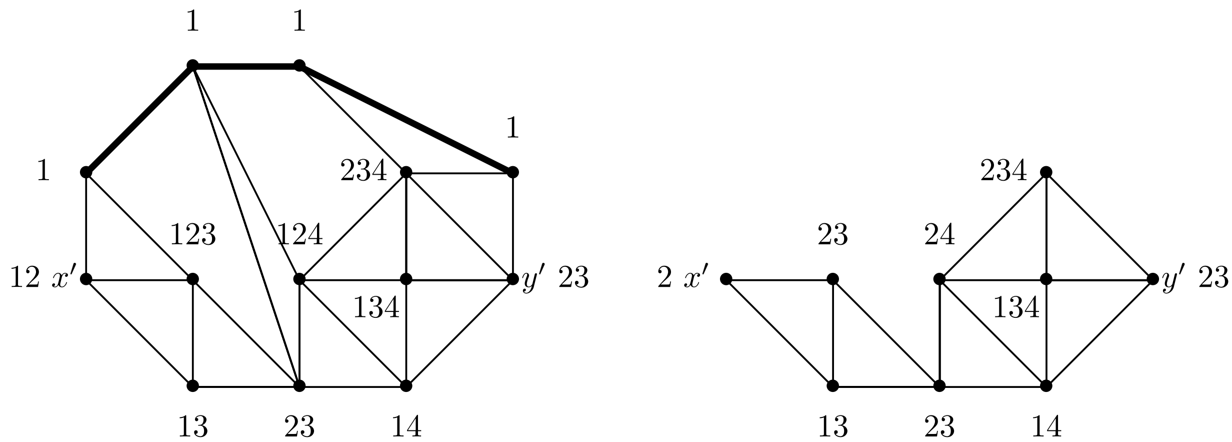

Figure 4.1

This means that v is on the outer face of H (necessarily between y' and x' in clockwise order, since P is precisely those vertices between y' and x' in counterclockwise order) and has a neighbor $w \in V(P)$ (between x' and y' in clockwise order). Then vw is a chord of H with x and y on opposite sides and v must separate x from y in H'. Hence $v \in S$, as claimed. □

Proof of Lemma 4.2: Let G, L, x, and y be as in the hypothesis. Since adding edges does not make G easier to color, we may assume that every bounded face is a triangle. Note that any edges added to triangulate bounded faces do not change whether a vertex is interior or not, so the hypotheses are still satisfied. We proceed by induction on the number of vertices in G. The statement is easy for graphs with at most three vertices. When G has more than three vertices, we have two cases:

Case 1: $S \neq \emptyset$. Let A be the block of G containing x. Let s be the vertex of S that is in A. Let B be the component of $G \setminus V(A)$ that contains y. Figure 4.2 depicts a typical example of case 1), with three components H_1, H_2, and B of $G \setminus V(A)$. In this example, S consists of the three vertices named s, s_2, s_3.

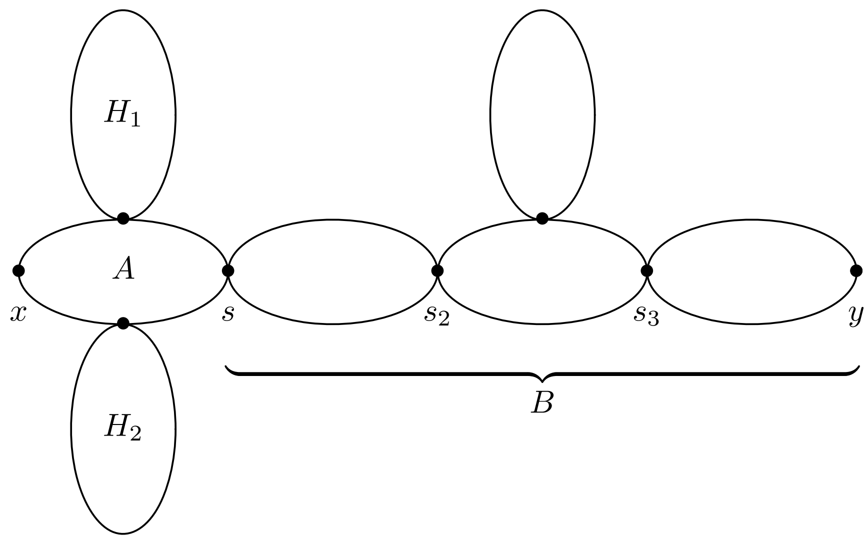

Figure 4.2

Color A by applying the induction hypothesis with s playing the role of y in the hypothesis. For each component H of $G \setminus V(A)$ other than B, Proposition 4.3 (applied with P being the one-vertex in A that has neighbors in H) assures us that we can apply the induction hypothesis to color H. To color B we replace s and apply the induction hypothesis with s playing the role of x in the hypothesis.

When we combine these colorings, we get a proper path-coloring of G. Different colorings are used on different pieces of G. These cannot interact except at the cut-vertices of G contained in A. For a cut-vertex v not equal to s, we applied Proposition 4.3 to remove the color used on v in A from the neighbors of v in other components of $G - V(A)$. For s, we note that Lemma 4.2 excludes any monochromatic paths through the distinguished vertices. This means that if s is part of a monochromatic path in each of A and B, it is an endpoint of both paths, and they can therefore be combined into one longer path.

Case 2: $S = \emptyset$. In this case, x and y are in the same block of G. Choose a color $c \in L(x)$, and choose a chordless path P in the following manner: if $x = y$ or $x \leftrightarrow y$ and $c \in L(y)$,

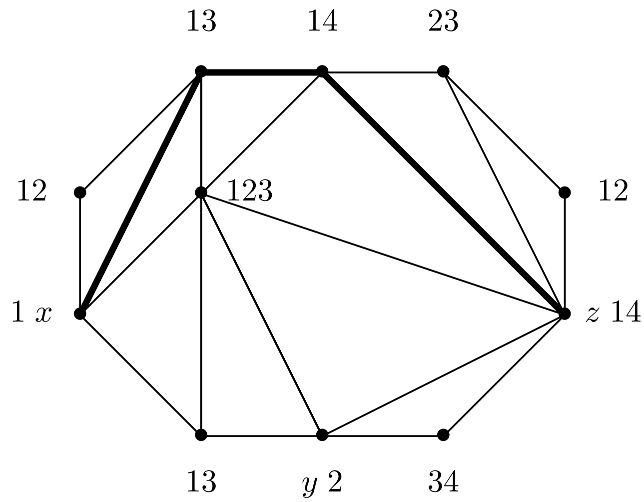

Figure 4.3

let P be the path consisting of x and y. Otherwise iteratively add vertices to the path P, beginning with x. Let v denote the current end of P. Let T be the set of vertices between v and y (including y) on the outer face that have c in their color list and are neighbors of v. If $T \neq \emptyset$, then append to P the vertex w of T that is closest to y on the outer face, and repeat with $v = w$. Otherwise, let P end at v. By construction, P is a chordless path. Figure 4.3 depicts an example of this case, with the resulting path P shown in bold.

Let z be the last vertex of P. Since P contains only vertices on the outer face of G, each edge of P is either a chord or is on the outer face. Each chord v_1v_2 in P defines a *lobe* of G, consisting of the cycle containing v_1 and v_2 and the vertices between them on the outer face, and the vertices interior to that cycle. Let the lobes of G defined in this manner be H_i, and let B be the subgraph induced by $V(P)$, the vertices from z to x (in clockwise order) on the outer face, and the vertices interior to the resulting cycle. Figure 4.4 depicts the subgraphs considered, for the example of Figure 4.3.

Now color all of $V(P)$ with c. For each lobe H_i, apply Proposition 4.3 with $P \cap H$

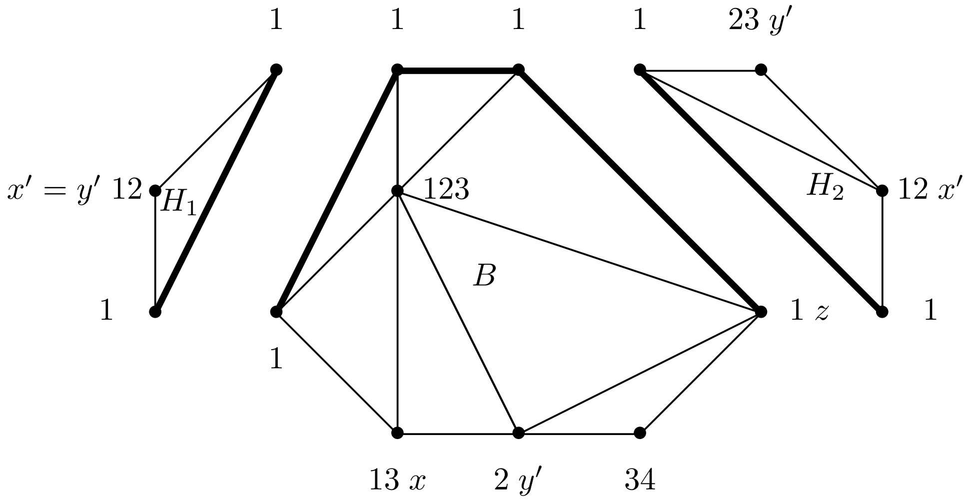

Figure 4.4

playing the role of P in the hypothesis, and then apply the induction hypothesis to the resulting H', L', x' and y' to get a proper path coloring. For B we must improve the argument of Proposition 4.3 slightly, since we must set $y' = y$ instead of setting it to be the next vertex after z. Otherwise, we may end up with $d_f(y) = 2$ after implying the induction hypothesis. The conclusion of Proposition 4.3 still holds if we set $y' = y$ since we have insured (by the choice of P) that no vertex on the outer face between z and y has c in its list, and thus that in L' those vertices still have list size at least 2. Therefore, we may also apply the induction hypothesis to B after removing P as in Proposition 4.3. As in the first case, these colorings form a proper path coloring of G. □

Note that Lemma 4.2 also proves directly that $\hat{\chi}_P(G) \leq 2$ if G is an outerplanar graph.

This proof yields an efficient algorithm for path 3-coloring. Jensen and Toft [14] note that Thomassen's proof [29] of the 5-choosability of planar graphs directly yields a simple linear 5-coloring algorithm, in contrast to the quadratic algorithms derived from traditional proofs of Heawood's theorem [12] that all planar graphs are 5-colorable. In a similar way, Lemma 4.2 yields a linear path 3-coloring algorithm for planar graphs, where previous proofs that planar graphs are path 3-colorable ([24] and [11]) lead to quadratic algorithms.

5. Spot colorings

In this chapter we consider a different generalized coloring parameter. The *spot chromatic number* of a graph, $\chi_S(G)$, is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of cliques. Unlike the other generalized coloring parameters we have seen, this parameter is not monotone. A subgraphs of a clique need not be a clique, so $\chi_S(H)$ may exceed $\chi_S(G)$ when H is a subgraph of G; for example, $\chi_S(P_3) = 2 > 1 = \chi_S(K_3)$.

Spot coloring was introduced by Grzegorz Kubicki [16] at a recent workshop on Discrete Mathematics in Louisville in June, 1997. He proved that if G is a complete multipartite graph with n vertices, then $\chi_S(G) \leq \min\{y : \binom{y}{2} \geq n\} - 1$. He also noted that if G spot k-chromatic, then $2G \vee K_1$ is spot $(k+1)$-chromatic. This graph is not spot k-colorable, since the vertex corresponding to K_1 would be the center vertex of an induced P_3 if it received any color used in both copies of G. Kubicki asked if all planar graphs are spot 3-colorable, and if there is a good upper bound for the spot chromatic number of an n vertex graph. We gave a negative answer to the first question, noting that the third iteration of Kubicki's construction that increases the spot chromatic number is still planar when we begin with $G_1 = K_1$ and set $G_i = 2G_{i-1} \vee K_1$ for $i > 1$. (see Figure 5.1).

5.1. Spot coloring the cartesian product of cliques

Another problem investigated during the workshop was to determine the spot chromatic number of the cartesian product of cliques. When G is the cartesian product of two cliques K_m and K_n, this can be restated as a matrix labeling problem. Indeed, the matrix labelling problem is perhaps even more natural than the spot coloring problem.

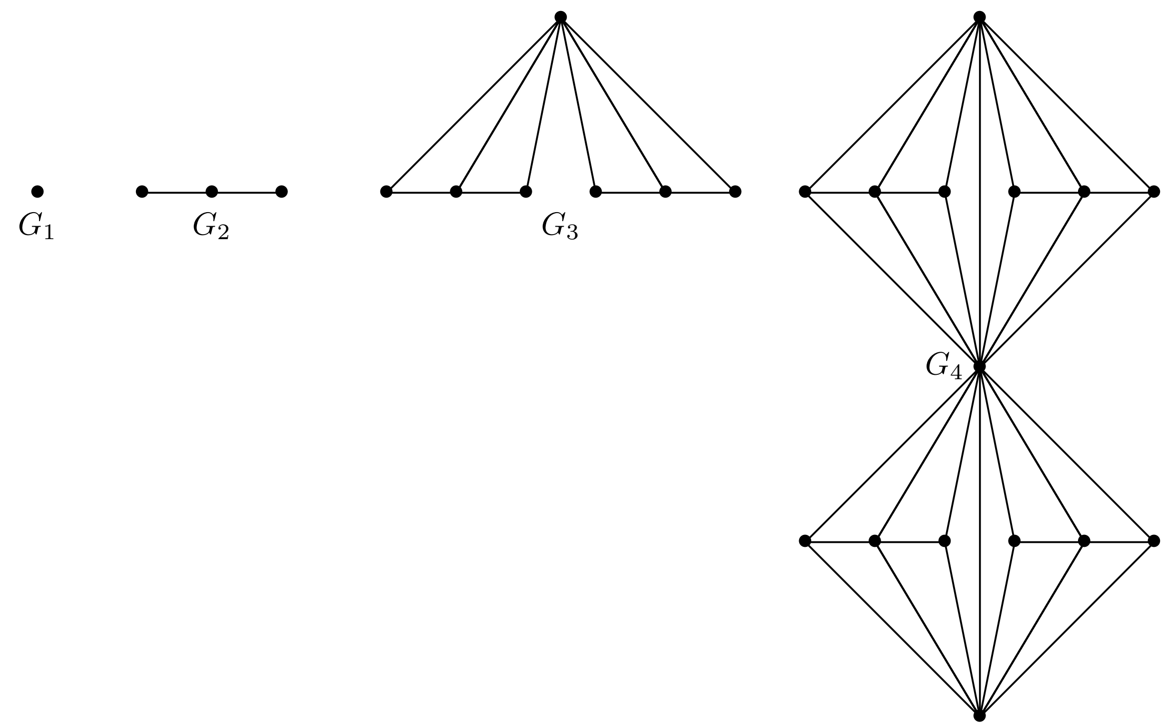

Figure 5.1

Proposition 5.1.1. *Represent $K_m \,\square\, K_n$ as a m by n matrix, where two elements are adjacent if they are in the same row or in the same column. A coloring of the elements of this matrix is a spot coloring of $K_m \,\square\, K_n$ if and only if no element has another element of the same color in both its row and its column.*

Proof: Such a configuration of three elements corresponds to an induced monochromatic P_3, but P_3 is not an induced subgraph of a disjoint union of cliques. Conversely, if no such configuration exists, then elements with a common color occur in groups, all in one row, or all in one column. Since no element appears in two such groups, the cliques corresponding to such groups are disjoint, and the color classes therefore induce disjoint unions of cliques. $\square$

It is easy to spot color $K_n \square K_n$ with n colors; simply color each row with a different color. The greedy coloring algorithm, where we iteratively choose a color class as large as possible among the uncolored vertices, does no better. It was briefly conjectured that $\chi_S(K_n \square K_n) = n$, but Chappell soon found a counterexample. He was able to spot color $K_9 \square K_9$ with only 8 colors. This was improved by West, who showed $\chi_S(K_6 \square K_6) = 5$. This yields $\chi_S(K_n \square K_n) \leq \frac{5n}{6} + O(1)$. Next, Jacobson [13] found a spot $(\frac{t}{2} + 2)$-coloring of $K_t \square K_t$ when t is even, and he proved that this is optimal.

Here we generalize Jacobson's coloring to show that $\chi_S(K_{mt} \square K_{nt}) \leq \frac{mnt}{m+n} + 2\min(m, n)$ whenever $m + n$ divides t. This is nearly optimal, in the sense that a spot coloring of $K_{mt} \square K_{nt}$ must use at least $\frac{mnt^2}{mt+nt-1}$ colors, and

$$\lim_{t\to\infty} \frac{\frac{mnt^2}{mt+nt-2}}{\frac{mnt}{m+n} + 2\min(m, n)} = 1.$$

We first give an upper bound on the size of a color class in a spot coloring of $K_m \square K_n$.

Proposition 5.1.2. *If m and n are at least 2, then no color class in a spot coloring of $K_m \square K_n$ can contain more than $m + n - 2$ vertices.*

Proof: Consider a color class C. Partition C into two sets A and B by placing a vertex v into A if there is another vertex of C in v's column. Otherwise, put v into B. No vertex in A has another vertex of the same color in its row, by Proposition 5.1.1 (see Figure 5.2). No vertex in B has another vertex of the same color in its column, by the choice of A. If A has vertices in c columns and B has vertices in r rows, then $|A| \leq m - r$ and $|B| \leq n - c$. If A or B is empty, then $|C| \leq \max\{m, n\}$, otherwise $|C| = |A| + |B| \leq m + n - (r + c) \leq m + n - 2$.

□

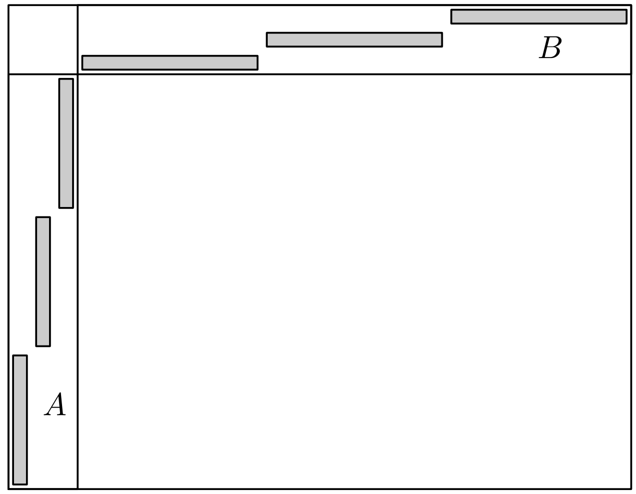

Figure 5.2

5.2. Jacobson's construction

Jacobson's construction of a spot $(\frac{t}{2}+2)$-coloring of $K_t \square K_t$ when t is even is illustrated in Figure 5.3. There are two types of color classes, "stairstep", represented by shades of gray, and "diagonal", represented by the numbers 1 and 2. All stairstep classes are isomorphic. Since each element of such a class has neighbors of the same class only in its row or only in its column, each such class induces a disjoint union of cliques. Also, classes 1 and 2 are independent sets, that is, tK_1.

5.3. The construction for $K_m \square K_n$

Theorem 5.3.1. *If $m+n$ divides t, then $\chi_S(K_{mt} \square K_{nt}) \leq \frac{mnt}{m+n} + 2\min(m,n)$*

Proof: Assume, without loss of generality, that $m \leq n$. As in Jacobson's construction, we use two types of color classes, "stairstep" and "diagonal" (see Figure 5.4). Each step in a stairstep class consists of a horizonal step of $\frac{nt}{m+n} - 1$ elements and a vertical strip of $\frac{mt}{m+n} - 1$ elements. The element to the right of the horizontal strip is the element below

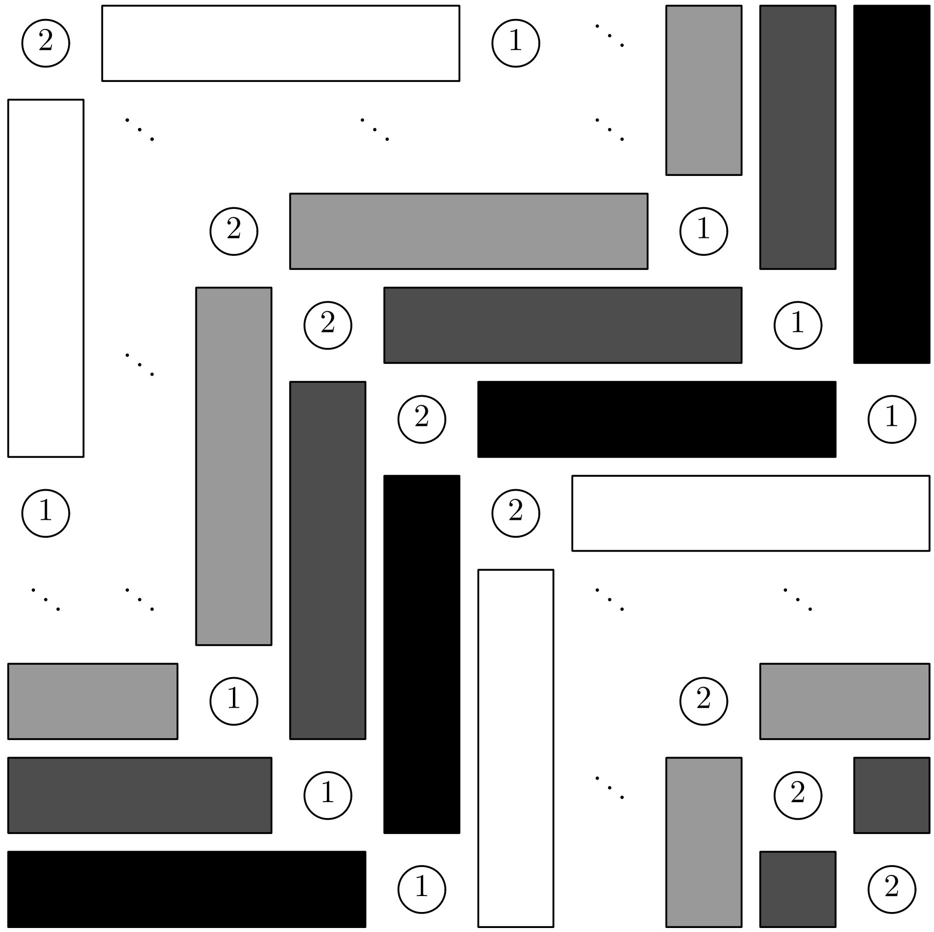

Figure 5.3

the vertical strip, so locally one diagonal is skipped. In each class there are $m+n$ steps. As in Jacobson's construction, each stairstep class induces a disjoint union of cliques, but we must show that they fit together correctly when shifted up and to the left, as shown in the diagram. We think of the matrix as occupying points (1,1) to (nt, mt) of the integer lattice. Consider the color B that starts a horizontal step in the lower left corner. The position cyclically to the left of this is $(nt, 1)$, the lower right corner. After $\frac{mnt}{m+n}$ shifts, it is at position $(nt - \frac{mnt}{m+n}, \frac{mnt}{m+n}+1)$. This is precisely one step above $(n\frac{nt}{n+m}, n\frac{mt}{n+m})$, which is the end of the nth vertical step in the original color class B, indicated in black. We have

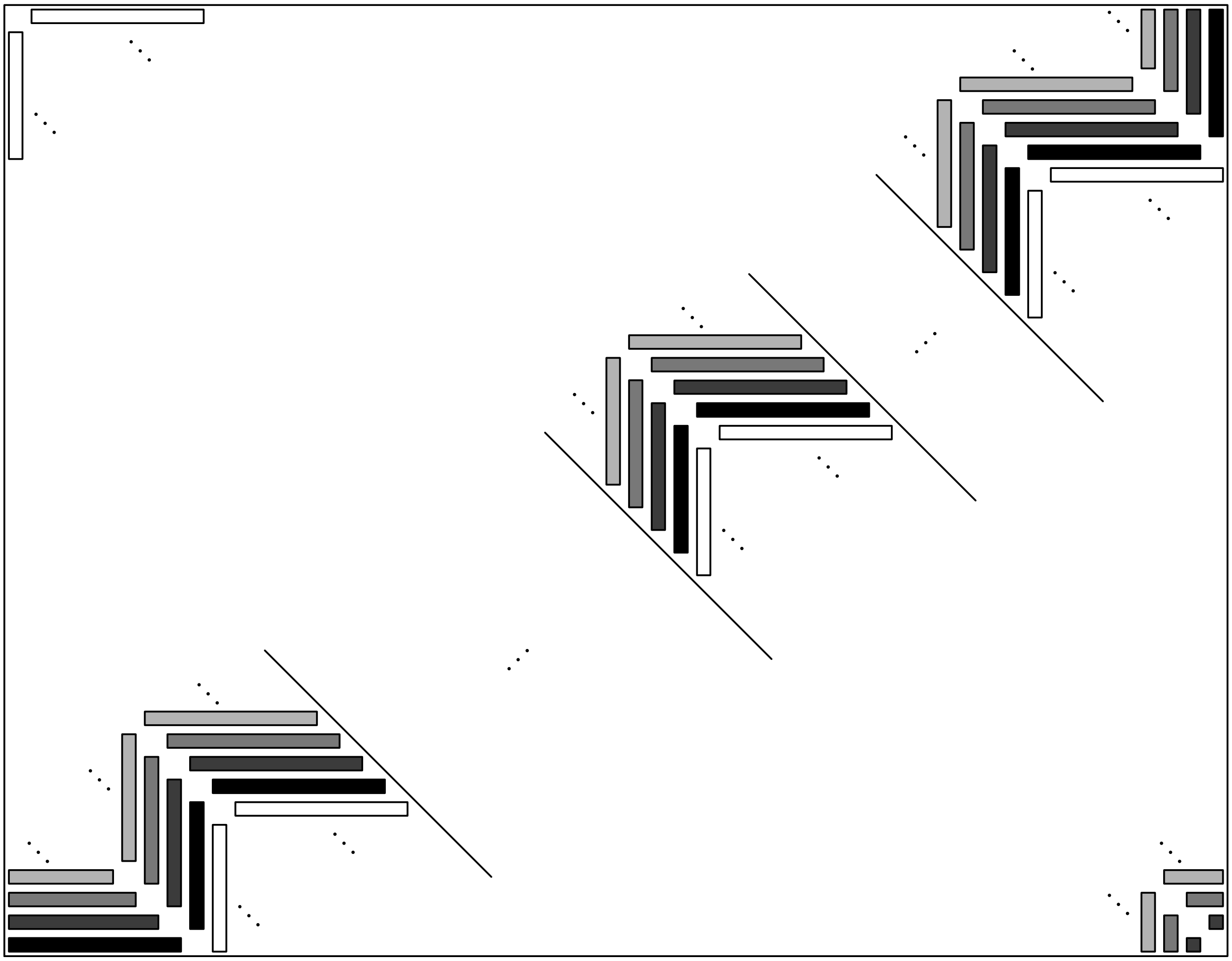

Figure 5.4

thus fit $\frac{mnt}{m+n}$ stairstep classes into the matrix without overlaps, but there are still elements of the matrix left uncolored. Examine a column of the partially colored matrix. There are repeated copies of the following formation: A vertical strip of $\frac{mt}{m+n} - 1$ elements of the same color, followed by an uncolored element, followed by $\frac{nt}{m+n} - 1$ elements each from a different color horizontal strip, followed by another uncolored element. Each formation uses $(\frac{mt}{m+n} - 1) + 1 + (\frac{nt}{m+n} - 1) + 1 = t$ elements. There are thus m copies of the formation in each column and therefore $2m$ uncolored elements in each column. We use an additional $2m$ color classes to color these uncolored elements, using every color class in each column.

Since each class is used only once in each column, it is used on a disjoint union of cliques. We have now used $\frac{mnt}{m+n} + 2m = \frac{mnt}{m+n} + 2\min(m,n)$ colors to spot color $K_m \square K_n$. $\square$

Theorem 5.3.1 is much better that the trivial bound $\chi_S(K_m \square K_n) \leq \min(m,n)$. For example, it shows that $\chi_S(K_{30} \square K_{60}) \leq 22$. By Proposition 5.1.2 at least $\lceil 1800/88 \rceil = 21$ colors are needed.

Proposition 5.1.2 shows that if we fix the ratio of the sizes of the cliques at $\frac{m}{n}$ but let their sizes go to infinity, our construction is nearly optimal, since

$$\lim_{t\to\infty} \frac{\frac{mnt^2}{mt+nt-2}}{\frac{mnt}{m+n} + 2\min(m,n)} = 1.$$

6. Observability and set-balanced k-edge-coloring

Černý, Horňák and Soták [5] studied a restricted form of edge-coloring. They defined the *observability* of a graph G, written $\mathrm{obs}(G)$, to be the least number of colors in a proper edge-coloring of G such that the color sets at vertices of G (sets of colors of their incident edges) are pairwise distinct. Here we introduce a generalization of observability.

A *set-balanced k-edge-coloring* of a graph G is a proper k-edge-coloring of G such that for each vertex degree d each d-set of colors is used about equally often. More precisely, each d-set appears at $\lfloor n_d/\binom{k}{d}\rfloor$ or $\lceil n_d/\binom{k}{d}\rceil$ vertices, where n_d is the number of vertices of degree d. For example, a d-regular graph G with n vertices has a set-balanced k-edge-coloring if the edges of G can be properly colored so that each set of d colors appears at $\lfloor n/\binom{k}{d}\rfloor$ or $\lceil n/\binom{k}{d}\rceil$ vertices. When k is large enough, the number of d-sets becomes so large that $\lfloor n_d/\binom{k}{d}\rfloor = 0$ for each d, and the question of whether G has a set-balanced k-edge-coloring becomes the question of whether the observability of G is at most k.

Černý *et. al* determined the observability of complete graphs, paths, cycles, wheels, and complete multipartite equipartite graphs. We generalize these results to determine exactly the values of k such that graphs from these classes have set-balanced k-edge-colorings. For many of these classes, set-balanced k-edge-colorings exist if and only if k is at least the observability. We also prove that certain 2-regular graphs with n vertices have observability equal to $\min\{j : \binom{j}{2} \geq n\}$. Horňák conjectured [21] that this true for all 2-regular graphs. This would say that the lower bound on observability arising from the trivial counting argument is the correct value for these graphs. Černý, Horňák and Soták [5] proved this for cycles, and Steiner triple systems provide examples for the special case in which G is a disjoint union of 3-cycles and n is congruent to 0 or 1 modulo 6.

6.1. Special classes

We first list trivial necessary and trivial sufficient conditions for the existence of set-balanced k-edge-colorings.

Lemma 6.1.1. *If G has a set-balanced k-edge-coloring, then $k \geq \chi'(G) \geq \Delta(G)$.*

If G is d-regular and $\chi'(G) = d$, then G has a set-balanced d-edge-coloring.

If $k \geq \mathrm{obs}(G)$, then G has a set-balanced k-edge-coloring.

Proof: A set-balanced k-edge-coloring must be a proper edge-coloring, so the first statement holds. In a proper d-edge-coloring of a d-regular graph the single d-set of colors appears at every vertex, so this is a set-balanced k-edge-coloring, and proves that the second statement holds. Any coloring showing $\mathrm{obs}(G) \leq k$ is a set-balanced k-edge-coloring of G, since those color sets that appear at vertices do so at most once. All others appear zero times, so the third statement follows. □

We begin with some easy examples.

Example. *W_n, the wheel with n vertices.*

Černý *et al.* [5] showed for $n \geq 5$ that the observability of the n vertex wheel $W_n = K_1 \vee C_{n-1}$ is $n-1$. Since $\Delta(W_n) = n-1$, Lemma 6.1.1 implies that W_n has a set-balanced k-edge-coloring if and only if $k \geq n-1$. □

Example *The clique K_n.*

Černý *et al.* [5] showed for $n \geq 3$ that $\mathrm{obs}(K_n) = 2\lceil \frac{n+1}{2} \rceil - 1$. When n is odd, $\chi'(K_n) = n$. Thus Lemma 6.1.1 implies, for odd n, that K_n has a set-balanced k-edge-coloring if and only if $k \geq n$. The even case is slightly more interesting. When n is even, K_n is $(n-1)$-edge-colorable (using rotations of Figure 6.1), so it has a set-balanced

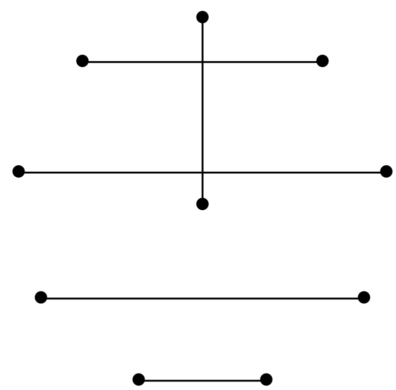

Figure 6.1

$(n-1)$-edge-coloring. The next proposition implies that it does not have a set-balanced n-edge-coloring. Thus, for even n, K_n has a set-balanced k-edge-coloring if and only if $k = n-1$ or $k \geq n+1$. □

Proposition 6.1.2. *Let G be a regular graph of degree d with n vertices. If $(d+1)$ does not divide n, then G does not have a set-balanced $(d+1)$-edge-coloring.*

Proof: Consider a set-balanced $(d+1)$-edge-coloring of G. For each color i, let α_i denote the number vertices at which i does not appear. Every α_i has the same parity as n, since each edge of color i is incident to exactly two vertices. However, the set of colors used at a vertex is identified by which color is missing, so the property of set-balancing requires that the numbers $\{\alpha_i\}$ differ by at most 1. Since every α_i has the same parity, we must actually have $\alpha_i = c$ for all i and some constant c. We then have $(d+1)c = n$. □

Černý *et. al* [5] made the observation that there is a bijection between edge-colorings of C_n that demonstrate $\text{obs}(C_n) \leq k$ and closed trails of length n in K_k. Similarly, there is a bijection between edge-colorings of P_n that demonstrate $\text{obs}(C_n) \leq k$ and non-closed trails of length n in K_k. We generalize these observations.

Proposition 6.1.3. *There is a bijection between set-balanced k-edge-colorings of C_n and closed walks of length n on K_k that use each edge $\lfloor n/\binom{k}{2} \rfloor$ or $\lceil n/\binom{k}{2} \rceil$ times.*

Proof: Edges of C_n correspond to the vertices of the walk, which denote colors. The edges of the walk can be viewed as pairs of successive colors on C_n. The requirement that every edge of K_k is used $\lfloor n/\binom{k}{2} \rfloor$ or $\lceil n/\binom{k}{2} \rceil$ times in the walk corresponds to the requirement that pairs of colors at the vertices of C_n must appear $\lfloor n/\binom{k}{2} \rfloor$ or $\lceil n/\binom{k}{2} \rceil$ times. □

To state exactly when C_n has a set-balanced k-edge-coloring, we also need the following simple proposition, proved in [5].

Proposition 6.1.4. *If k is odd, then K_k has a closed trail of length l if and only if $3 \leq l \leq \binom{k}{2} - 3$ or $l = \binom{k}{2}$. If k is even, then K_k has a closed trail of length l if and only if $3 \leq l \leq \binom{k}{2} - \frac{k}{2}$.*

Because there are $\binom{k}{2}$ 2-sets from a set of k colors, a set-balanced k-edge-coloring of a 2-regular graph of order n must use each 2-set at $\lfloor n/\binom{k}{2} \rfloor$ or $\lfloor n/\binom{k}{2} \rfloor + 1$ vertices. Note that q in the next theorem is $\lceil n/\binom{k}{2} \rceil - 1$.

Theorem 6.1.5. *Let G be a cycle of length n. Write $n = q\binom{k}{2} + r$, where q and r are integers and $1 \leq r \leq \binom{k}{2}$. Then G has a set-balanced k-edge-coloring if and only if:*

1) k is odd and $r = \binom{k}{2}$ or $3 \leq r \leq \binom{k}{2} - 3$,

2) k is even, q is even, and $3 \leq r \leq \binom{k}{2} - \frac{k}{2}$, or

3) k is even, q is odd, and $\frac{k}{2} \leq r \leq \binom{k}{2} - 3$.

Proof: In each case listed, we show that there is a connected even multigraph on k vertices with n edges in which each of the $\binom{k}{2}$ edges has multiplicity q or $q+1$. Every connected even multigraph has a closed Eulerian trail, which corresponds to a closed walk in K_k using each edge q or $q+1$ times. By Proposition 6.1.3 this yields a set-balanced k-edge-coloring of G.

In cases 1 and 2, Proposition 6.1.4 guarantees the existence of a closed trail T of length r. We begin with q copies of each edge and add an additional copy of each edge in T. Each vertex then has $q(k-1)$ edges, plus some additional edges from T. Since T passes through each vertex an even number of times, and $q(k-1)$ is even, this is an even graph. When $q \geq 1$ it is connected, since there is at least one copy of every edge, and when $q = 0$ it is exactly T and thus is still connected.

In case 3, Proposition 6.1.4 guarantees the existence of a closed trail T of length $\binom{k}{2} - r$. We again begin with q copies of each edge. In this case, however, we add an additional copy of the r edges not used in T to reach a total of n edges. Since k is even, each vertex is incident to an odd number of edges not on T. Also, $q(k-1)$ is odd, so each vertex again has even total degree. Since q is odd, there is at least one copy of every edge, so the graph is connected.

To prove that the condition is necessary, note that if there is a set-balanced k-edge-coloring of C_n, then Proposition 6.1.3 yields a closed walk of length n on K_k with every edge used q or $q+1$ times. Deleting q copies of each edge leaves a graph G with r edges on k vertices.

If k is odd or q is even then G is an even graph. Choosing one vertex from each component of G and merging them into a single vertex inheriting all the incident edges leaves us with a connected even graph with r edges on no more than k vertices. Since such a graph has a (closed) Eulerian trail of length r, Proposition 6.1.4 implies that r is as specified.

If k is even and q is odd, then G is not an even graph, but $\overline{G}$ is then an even graph having $\binom{k}{2} - r$ edges. Merging one chosen vertex from each component of $\overline{G}$ leaves us with a connected even graph. Since such a graph has a (closed) Eulerian trail, Proposition 6.1.4 again implies that r is as specified. □

A statement similar to Proposition 6.1.3 can be made about paths P_n, but we must specify non-closed walks so that the edges at the ends of the path receive different colors. Since P_n has $n-1$ edges, we must find non-closed walks with $n-1$ vertices, that is, walks of length $n-2$.

Proposition 6.1.6. *There is a bijection between set-balanced k-edge-colorings of P_n and non-closed walks of length $n-2$ on K_k that use each edge $\lfloor n/\binom{k}{2} \rfloor$ or $\lceil n/\binom{k}{2} \rceil$ times.*

Proof: Edges of P_n correspond to the vertices of the walk, which denote colors. Edges of the walk can be viewed as pairs of successive colors on P_n. Endpoints of the walk correspond to the pendant edges of P_n. The requirement that every edge of K_k is used c or $c+1$ times in the walk corresponds to the requirement that each 2-set of colors appears at $\lfloor n/\binom{k}{2} \rfloor$ or $\lceil n/\binom{k}{2} \rceil$ of the degree 2 vertices of P_n. As noted, the requirement that the trail be non-closed corresponds to the requirement that the two edges at the ends of P_n get different colors. □

Again, we need a simple proposition from [5].

Proposition 6.1.7. *If k is odd, then K_k has a non-closed trail of length l if and only if $l \in [1, \binom{k}{2} - 1]$. If k is even, then K_k has a closed trail of length l if and only if $l \in [1, \binom{k}{2} - \frac{k}{2} + 1]$.*

Theorem 6.1.8. *Let G be a path of length n. Write $n - 2 = q\binom{k}{2} + r$, where q and r are integers and $1 \le r \le \binom{k}{2}$. Then G has a set-balanced k-edge-coloring if and only if:*

1) k is odd and $1 \le r \le \binom{k}{2} - 1$,

2) k is even, q is even, and $1 \le r \le \binom{k}{2} - \frac{k}{2} + 1$, or

3) k is even, q is odd, and $\frac{k}{2} - 1 \le r \le \binom{k}{2} - 1$.

Proof: In all cases we show that there is a connected multigraph G on k vertices with $n - 2$ edges such that 1) each of the $\binom{k}{2}$ edges has multiplicity q or $q + 1$, and 2) exactly two vertices of G have odd degree. Every such multigraph has an Eulerian trail of length $n - 2$ beginning at one vertex with odd degree and ending at the other, and this yeilds (by Proposition 6.1.3) a set-balanced k-edge-coloring of G.

In cases 1 and 2, Proposition 6.1.7 guarantees the existence of a non-closed trail T of length r. We begin with q copies of each edge and add an additional copy of each edge in T. Each vertex then has $q(k-1)$ edges, plus some additional edges from T. Since T passes through all but two vertices an even number of times, and $q(k - 1)$ is even, all but two vertices of this multigraph have even degree. When $q \ge 1$ it is connected, since there is at least one copy of every edge, and when $q = 0$ it is exactly T and thus is still connected.

In case 3, Proposition 6.1.7 guarantees the existence of a non-closed trail T of length $\binom{k}{2} - r$. We again begin with q copies of each edge. In this case, however, we add an additional copy of the r edges not used in T to reach a total of n edges. Since k is even, all but two vertices are incident to an odd number of edges not on T. Also, $q(k - 1)$ is odd, so again all but two vertices of this multigraph have even degree. Since q is odd, there is at least one copy of every edge, so the graph is connected.

To prove that the condition is necessary, note that if there is a set-balanced k-edge-coloring of P_n, then Proposition 6.1.6 yields a non-closed walk of length $n-2$ on K_k with every edge used q or $q+1$ times. Deleting q copies of each edge leaves a graph G with r edges on k vertices.

If k is odd or q is even, then G has exactly two vertices with odd degree. Note that the two vertices with odd degree must be in the same component. Thus, choosing one vertex from each component of G and merging them into a single vertex inheriting all the incident edges leaves us with a connected graph with r edges on no more than k vertices where exactly two vertices have odd degree. Since such a graph has a non-closed Eulerian trail of length r, Proposition 6.1.7 implies that r is as specified.

If k is even and q is odd, then G has exactly two vertices with even degree, but $\overline{G}$ (which has $\binom{k}{2}-r$ edges) is then a graph with exactly two vertices of odd degree. Merging one chosen vertex from each component of $\overline{G}$ leaves us with a connected graph where all but two vertices have even degree. Since such a graph has a non-closed Eulerian trail, Proposition 6.1.7 again implies that r is as specified. □

Let $K_{p\times q}$ denote the complete multipartite graph with p parts of q vertices. These are precisely the regular complete multipartite graphs, with degree $q(p-1)$. Horňák and Soták [22] proved that $\mathrm{obs}(K_{p\times q}) = q(p-1)+2$. Lemma 6.1.1 thus implies that $K_{p\times q}$ has a set-balanced k-edge-coloring for all $k \geq q(p-1)+2$ and for no $k < q(p-1) = \Delta(K_{p\times q})$. The following two propositions complete the analysis by determining when $K_{p\times q}$ has set-balanced k-edge-colorings for $k \in \{q(p-1), (q(p-1)+1\}$. We write $G[H]$ to denote the graph formed by replacing every vertex of G by a copy of H with the same neighborhood. Note that $K_{p\times q}$ is exactly $K_p[qK_1]$.

Proposition 6.1.9. *The complete multipartite graph $K_{p\times q}$ can be properly $q(p-1)$-edge-colored if and only if pq is even.*

Proof: We repeatedly use the fact that if G_1 can be decomposed into copies of G_2 and G_2 can be decomposed into copies of G_3 then G_1 can be decomposed into copies of G_3. Also, if G decomposes into copies of F, then $G[H]$ decomposes into copies of $F[H]$.

If p is even, consider $K_{p\times q}$ as $K_p[qK_1]$. Since K_p has a decomposition into perfect matchings, $K_{p\times q}$ has a decomposition into copies of $\frac{p}{2}K_2[qK_1]$, which equals $\frac{p}{2}K_{q,q}$. Since complete bipartite graphs have decompositions into perfect matchings, so do disjoint unions of such graphs. Transitivity of decomposition now completes the proof.

If p is odd but q is even, then we reduce the problem to providing a decomposition of $K_{p\times 2}$ into perfect matchings. If we split each partite set of $K_{p\times q}$ into two equal size parts, and collapse the complete bipartite graphs occuring between these new parts into edges, we obtain $K_{p\times 2}$. In other words, $K_{p\times q} = K_{p\times 2}[\frac{q}{2}K_1]$. If we can find a decomposition of $K_{p\times 2}$ into perfect matchings, then $K_{p\times q}$ decomposes into copies of $pK_2[\frac{q}{2}K_1]]$ which equals $pK_{\frac{q}{2},\frac{q}{2}}$. Decomposition of $K_{\frac{q}{2},\frac{q}{2}}$ into matchings and transitivity of decomposition then complete the proof.

It is well known that K_p has a decomposition into spanning cycles when p is odd, as illustrated by rotations of Figure 6.2. Thus $K_{p\times q} = K_p[2K_1]$ can be decomposed into copies of $C_p[2K_1]$. If we represent $C_p[2K_1]$ with vertices $c_1, c_2, \ldots, c_p$ and $c'_1, c'_2, \ldots, c'_p$ where indices are treated modulo p, then we have edges $c_i \leftrightarrow c_{i+1}, c_i \leftrightarrow c'_{i+1}, c'_i \leftrightarrow c_{i+1}$, and $c'_i \leftrightarrow c'_{i+1}$. The cycles $(c_1, c_2, \ldots, c_{p-1}, c_p, c'_{p-1}, c'_{p-2}, \ldots, c'_1, c'_p)$ and $(c_1, c'_2, c_3, \ldots, c'_{p-1}, c'_p, c_{p-1},$

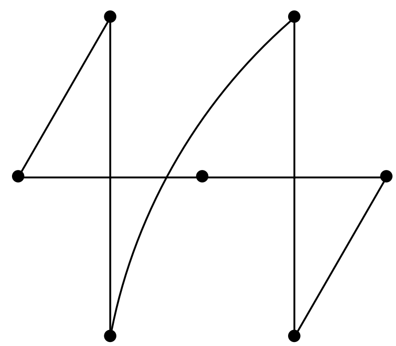

Figure 6.2.

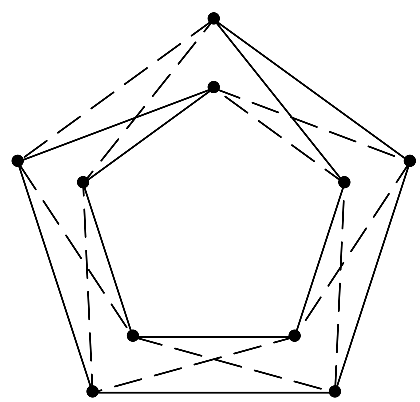

Figure 6.3.

$c'_{p-2}, \ldots, c_2, c'_1, c_p)$ form a decomposition of $C_p[2K_1]$ into two even cycles, as shown in Figure 6.3. Since even cycles have a decomposition into perfect matchings, transitivity of decomposition again completes the proof. □

Proposition 6.1.10. *The complete multipartite graph $K_{p\times q}$ has a set-balanced $(q(p-1)+1)$-edge-coloring if and only if $p=1$ or $q=1$ and p is odd.*

Proof: When $p=1$ the graph has no edges, and the statement holds trivially. When

$q = 1$ the graph is a clique, and the result has already been shown. When p and q are both at least 2, we apply Proposition 6.1.2. It suffices to show that $(p-1)q+1$ divides pq only if $p = 1$ or $q = 1$. If $(pq - q + 1)$ divides pq, then $pq/(pq - q + 1)$ is an integer, but then $(q-1)/(pq - q + 1)$ is also an integer. Thus $q - 1 = 0$ or $q - 1 \geq pq - q + 1$; the latter implies $p \leq (2 - 2/q)$, which contradicts $p \geq 2$. By Proposition 6.1.2, $K_{p \times q}$ thus has no $((p-1)q+1)$-edge-coloring. □

6.2. Observability of regular graphs

Černý, Horňák and Soták [5] conjectured the following:

Conjecture 6.2.1. *If* $\binom{k-1}{d-1}$ *is even, then* $\text{obs}(G) = k$ *for every* d*-regular graph with* $\binom{k}{d}$ *vertices.*

The converse is easy. Since G has $\binom{k}{d}$ vertices, we require each d-set of colors to appear at exactly one vertex. Thus, each color appears at $\binom{k-1}{d-1}$ vertices. This must be even, since each edge is incident to two vertices. We propose a generalization, motivated by the following example.

Example. *The cube* Q_3. The cube Q_3 has observability 6, although $\binom{5}{3} = 10 > 8$. Horňák originally showed this by case analysis [23], but we give a short argument for the more general statement that no 3-regular 8-vertex graph has observability 5. Suppose some such graph has a set-balanced 5-edge-coloring. Each color appears 6 times in the ten 3-color subsets of five colors. If we choose two 3-sets to leave out, then at least one color in each of those 3-sets will not be in the other. These colors then appear exactly 5 times in the 8 remaining subsets, but as noted above, each color appears at an even number of vertices in every edge-coloring of a graph. □

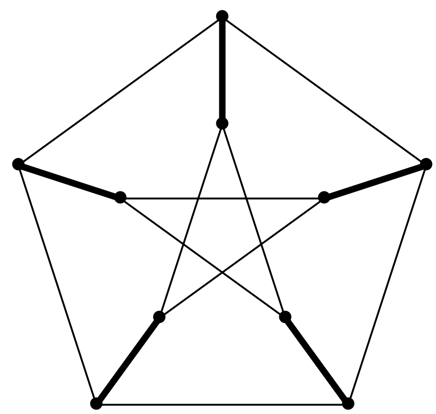

Figure 6.4.

Conjecture 6.2.2. *Let G be a d-regular graph with n vertices, and let k be the smallest integer such that $\binom{k}{d} \geq n$. Then $\text{obs}(G) = k$ if and only if there exists a family of n d-subsets of $[k]$ such that each color appears in an even number of sets in the family.*

As in Conjecture 6.2.1, the condition is necessary for $\text{obs}(G) = k$, since we must be able to choose color sets for the vertices such that each color is used at an even number of vertices.

If Conjecture 6.2.2 is true, then determining the observability of a regular graph is actually a problem in design theory, and the structure of the graph plays no part. The obvious extension of the conjecture to set-balanced k-edge-colorings is false. If we drop the condition that $\binom{k}{d} \geq n$ and require a set-balanced k-edge-coloring of G rather than $\text{obs}(G) = k$, then every regular graph with edge-chromatic number greater than the degree is a counterexample. For example, consider the Peterson graph, shown in Figure 6.4. In a 3-edge-coloring of this graph, each color class must be a perfect matching, but if any perfect matching is removed, the remainder consists of two 5-cycles, and is not 2-edge-colorable.

The results of Černý *et. al* on the observability of cycles verify that the conjecture is true for connected graphs that are 2-regular. There are few further results, even for the seemingly easy case of Conjecture 6.2.1 for disconnected 2-regular graphs. For example, let G be a graph consisting of m disjoint cycles with lengths $\alpha_1, \alpha_2, \ldots, \alpha_m$, where $\sum_{i=1}^{m} \alpha_i = \binom{k}{2}$. The techniques of Section 6.1 allow us to restate the problem: it suffices to decompose K_k into closed trails of lengths $\alpha_1, \alpha_2, \ldots, \alpha_m$. This can only be done when k is odd, since when k is even, each vertex of K_k has odd degree. This problem has been well studied in the literature when the closed trails are all required to be cycles of the same length; the survey article by Lindner and Rodger [18] lists many results.

Example. *Steiner triple systems.* A Steiner triple systems on n elements is a set of triples from $[n]$ such that each pair of elements appears together in exactly one triple. Steiner triple systems on n elements exist whenever $n \equiv 1 \pmod 6$ or $n \equiv 3 \pmod 6$. We can find a decomposition of K_n into 3-cycles whenever a Steiner triple system on n elements exists. Each triple yields a 3-cycle on the vertices of K_n, and each edge of K_n then occurs in one triple. □

We provide theorems to settle two other cases of Conjecture 6.2.1. The first is suggested by the well-known fact that K_n can be decomposed into paths of lengths $1, 2, \ldots, n-1$. When n is odd, an Eulerian circuit of K_n yields a decomposition of K_n into trails of arbitrary lengths, but it does not yield closed trails of those lengths.

Theorem 6.2.3. *For odd $n \geq 3$, let $\alpha_1 = 3$ and $\alpha_i = i + 1$ for $2 \leq i \leq n-2$. Then K_n can be decomposed into pairwise edge-disjoint closed trails of lengths $\alpha_1, \alpha_2, \ldots, \alpha_{n-2}$.*

Proof: We inductively form a family $\mathcal{F}_n$ consisting of $n-3$ closed trails with lengths $\alpha_2, \alpha_3, \ldots, \alpha_{n-2}$ such that $\mathcal{F}_n$ together with the cycle $(1, n, n-1)$ uses every edge of K_n

exactly once. Since $(1, n, n-1)$ has length $\alpha_1 = 3$, this is the required decomposition.

We begin with $\mathcal{F}_3 = \emptyset$, since the cycle (1,3,2) uses every edge of K_3. Suppose $n > 3$. Given the family $\mathcal{F}_{n-2}$ we specify two additional closed trails of lengths $n-2$ and $n-1$. We use all the new edges between $\{n-1, n\}$ and $\{2, ..., n-2\}$; there are $2n-6$ of these. We also use the edges of the previous special cycle $(1, n-2, n-3)$; these bring the total to $2n-3$ edges, as desired, and this cycle will be replaced by $(1, n, n-1)$. Because of these three special edges, we handle $n-2$ and $n-3$ specially and define $S = \{2, ..., n-4\}$.

Each element of S occurs in exactly one of the two trails of lengths $n-2$ and $n-1$, occuring once between n and $n-1$ to contribute its two edges to those vertices. We denote the elements of S by x's below; their order is unimportant. There are $(n-5)/2$ such elements in each trail. When $n \equiv 1 \pmod 4$, the number of x's in each trail is even, so the edges from n to $\{n-3, n-2\}$ occur in the same trail, and similarly for $n-1$. When $n \equiv 3 \pmod 4$, the number of x's in each trail is odd, so the edges from n to $\{n-3, n-2\}$ occur in different trails and similarly for $n-1$. In each case, these two trails complete the required decomposition.

When $n \equiv 1 \pmod 4$, the two closed trails are:

$$(n-3, n-2, n, \overbrace{x, n-1, x, n, \ldots, x, n-1, x, n}^{(n-5)/4 \text{ groups}})$$

$$(n-2, 1, n-3, n-1, \overbrace{x, n, x, n-1, \ldots, x, n, x, n-1}^{(n-5)/4 \text{ groups}}).$$

When $n \equiv 3 \pmod 4$, the two closed trails are:

$(n-7)/4$ groups

$$(n-3, n-2, n, x, n-1, \overbrace{x, n, x, n-1}, \ldots, \overbrace{x, n, x, n-1})$$

$$\overbrace{}^{(n-7)/4 \text{ groups}}$$

$$(n-2, 1, n-3, n, x, n-1, \overbrace{x, n-1, x, n}, \ldots, \overbrace{x, n-1, x, n}).$$

□

Our next theorem about decompositions is motivated by the example of Steiner triple systems. We wish to show that whenever n is odd, the graph on $\binom{n}{2}$ vertices consisting of as many cycles as possible has observability n. Steiner triple systems decompose K_n into triangles when $n \equiv 1$ or $3 \pmod 6$. When $n \equiv 5 \pmod 6$, this is impossible since $\binom{n}{2} \equiv 1 \pmod 3$. In this case we show how to decompose K_n into triangles and one four-cycle. We will need the following theorem of Baker [4]. We will use only the case of the theorem where $6a+3$ and $a+k$ are relatively prime, in which case the decomposition consists of triangles and a single spanning cycle.

Theorem 6.2.4. *The clique K_{6a+3} can be decomposed into triangles and the disjoint union of $\gcd(6a+3, a+k)$ cycles of length $(6a+3)/\gcd(6a+3, a+k)$ where $1 \le k \le 2a+1$ if either*

1) k is odd and $a \equiv 0$ or $1 \pmod 4$, or

2) k is even and $a \equiv 2$ or $3 \pmod 4$.

Lemma 6.2.5. *For each positive integer a, there exists a positive integer k fulfilling the hypothesis of Theorem 6.2.4 such that $6a+3$ and $a+k$ are relatively prime.*

Proof: When $a \equiv 2 \pmod 4$, simply take k so that $a+k$ is a power of 2 and $a+1 \le a+k \le 3a+1$. When $a \equiv 3 \pmod 4$ take $k = \frac{a+1}{2}$ so that $\gcd(6a+3, a+k) =$

$\gcd(6a+3, \frac{3a+1}{2}) = 1$ since $(6a+3) - 4\frac{3a+1}{2} = 1$. When $a \equiv 0$ or $1 \pmod 4$ let $k = 2a+1$ so that $\gcd(6a+3, a+k) = \gcd(6a+3, 3a+1) = 1$ since $(6a+3) - 2(3a+1) = 1$. □

Theorem 6.2.6. *If $n \equiv 5 \pmod 6$, then K_n can be decomposed using triangles and one 4-cycle.*

Proof: Decompose K_{n-2} into triangles and a single $(n-2)$-cycle using Lemma 6.2.5 and Theorem 6.2.4. Let the $(n-2)$-cycle be $(v_1, v_2, \ldots, v_{n-2})$. Introduce two new vertices x and y and form a decomposition of K_n using the triangles from the decomposition of K_{n-2} and the triangles (x, v_{2i+1}, v_{2i+2}) and (y, v_{2i+2}, v_{2i+3}) for $0 \leq i \leq \frac{n-5}{2}$. The 4-cycle (x, v_{n-2}, v_1, y) completes the decomposition.

Note that we have constructed k to be odd when $a \equiv 0$ or $1 \pmod 4$ and even when $a \equiv 2$ or $3 \pmod 4$. □

7. Twisted Hypercubes

As described in Chapter 1, we seek k-regular graphs of order 2^k with small diameter. We consider the "twisted hypercubes", defined recursively by Slater [28] (and independently others) as follows. The 1-vertex graph K_1 is a twisted hypercube of dimension 0. To construct a *twisted hypercube* of *dimension* $k \geq 1$, take any two twisted hypercubes of dimension $k-1$ and connect them with any matching. We define $\mathcal{G}_k$ to be the set of all hypercubes of dimension k. The classical k-dimensional hypercube Q_k belongs to $\mathcal{G}_k$ according to its usual construction. Let $Q_0 = K_1$. For $k \geq 1$, construct Q_k by connecting two copies of Q_{k-1} using the matching that connects the two copies of each vertex.

In this chapter we describe a construction that, given a twisted hypercube of dimension k and radius r, creates a twisted hypercube of dimension $2k+r$ and radius no more than $2r$. By iterating this construction, we obtain twisted hypercubes of dimension k and diameter approximately $4\frac{k}{\lg k}$. This is within a constant of best possible, since $\frac{k}{\lg k}$ is obtainable as a lower bound by a simple counting argument.

Figure 7.1 contrasts T_1, a twisted hypercube of dimension 3, with Q_3, the classical hypercube of dimension 3. When constructing the twisted hypercube of dimension 2 that forms the back face of T_1, note that we did not connect 100 to 110 and 101 to 111 as we must when constructing a classical hypercube. Instead, we have joined 100 to 111 and 101 to 110. Because of this "twist" the diameter of T_1 is 2, while Q_3 has diameter 3.

We first show that the radius (and thus the diameter) of any k-regular graph with 2^k vertices is at least $\frac{k}{\lg k} - 1$. We simply count the number of vertices a particular vertex v can reach in i steps. The only vertex that has distance 0 from v is v. There are k neighbors of v, these all have distance 1 from v. Each of these k vertices is adjacent to k vertices, thus

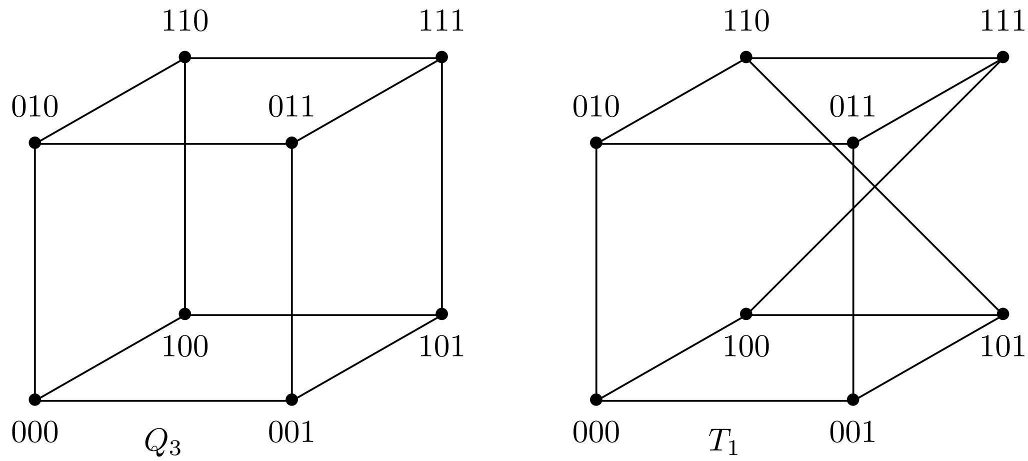

Figure 7.1

there are at most k^2 vertices that have distance 2 from v. In general, there are no more than k^d vertices at distance d from v, and thus no more than $1+k+k^2+\cdots+k^d = \frac{k^{d+1}-1}{k-1}$ having distance at most d from v. If a graph with 2^k vertices has diameter d, then we have $2^k \leq \frac{k^{d+1}-1}{k-1}$. We thus have $2^k(k-1) < k^{d+1}$ so $d \geq \frac{k+\lg(k-1)}{\lg k} - 1 \geq \frac{k}{\lg k} - 1$.

We next develop the notation we use to specify twisted hypercubes.

We use the set $\{0,1\}^k$ of binary vectors of length k as the vertex set of a twisted hypercube G of dimension k. Suppose that G is formed from two twisted hypercubes, say G_1 and G_2, of dimension $k-1$. We get vertex labels for G by prepending a 0 to the vertices of G_1, and prepending a 1 to the vertices of G_2. We must describe the edges of the matching connecting G_1 and G_2. Each such edge connects vertices with different leftmost bits. As we travel across the matching (say, from G_1 to G_2) we induce a permutation on the vertices of the $k-1$ dimensional hypercubes, that is, on the vectors with $k-1$ bits. This permutation is part of the specification of G.

Now consider how G_2 was constructed. Two twisted hypercubes of dimension $k-2$, say, H_1 and H_2, were connected by a matching. To describe G, we must also describe the matching that was used to connect H_1 and H_2. Edges from this matching must stay within G_2, that is, they can not change the leftmost bit. They must, however, change the second bit, this corresponds to moving from H_1 to H_2. The matching will thus be described by fixing the first bit, toggling the second bit, and performing a permutation on the vectors described by the remaining $k-2$ bits. Note that this permutation can be different in G_1 and G_2, that is, that the permutations described by matchings that fix the first bit and toggle the second bit can depend on the first bit.

We are now ready to give a precise definition. We will denote concatenation of vectors by ":" and length (number of coordinates) by $l()$. Given a vector $\vec{v} = (v_1, v_2, \ldots, v_k)$, we define $\vec{v}_{i,j} = (v_i, v_{i+1}, \ldots, v_j)$ for $1 \leq i \leq j \leq k$. Edges of G corresponding to a matching connecting two subgraphs of dimension i are described by associated permutations $\sigma^G_{\vec{a}}$, where $l(\vec{a}) = k-1-i$ and $\sigma^G_{\vec{a}}$ is a permutation of the binary vectors with length i. The 2^{k-1-i} different vectors of length $k-1-i$ correspond to the $2^{k-(i+1)}$ different subgraphs of dimension $i+1$. In each such subgraph we may choose a different matching connecting the two subgraphs of dimension i; $\sigma^G_{\vec{a}}$ is the permutation induced by that matching. Given a vertex $\vec{v}$ of G and a dimension j with $(1 \leq j \leq k)$, define $f_{G,j}(\vec{v})$ (the neighbor of $\vec{v}$ along the jth incident edge) as follows:

$$f_{G,j}(\vec{v}) = \begin{cases} \vec{v}_{1,j-1} : 1 : \sigma^G_{\vec{v}_{1,j-1}}(\vec{v}_{j+1,k}) & \text{if } v_j = 0, \text{ or} \\ \vec{v}_{1,j-1} : 0 : (\sigma^G_{\vec{v}_{1,j-1}})^{-1}(\vec{v}_{j+1,k}) & \text{if } v_j = 1. \end{cases}$$

Note that $f_{G,j}(\vec{v})$ always changes the jth bit of $\vec{v}$. In what follows we will always choose σ to be an involution, so that $f_{G,j}(\vec{v}) = \vec{v}_{1,j-1} : (1-v_j) : \sigma^G_{\vec{v}_{1,j-1}}(\vec{v}_{j+1,k})$. The complete specification of G consists of a permutation $\sigma^G_{\vec{u}}$ for each $\vec{u}$ with $0 \leq l(\vec{u}) < k-1$; the permutation describes the mates in dimension $l(\vec{u})+1$ for vertices with prefix $\vec{u}$.

Example. *The standard hypercube Q_k.* The standard hypercube Q_k is the twisted hypercube of dimension k with associated permutations $\sigma_{\vec{a}}^{Q_k}$ equal to the identity permutation for all $\vec{a}$. □

For any k and any i with $0 \leq i \leq k$, define $\tau_i(v_1, v_2, \ldots, v_k) = \vec{v}_{1,i-1} : (1 - v_i) : \vec{v}_{i+1,k}$. In other words, τ_i is the mapping that toggles the ith bit of $\vec{v}$. Also, let $\tau_0(\vec{v}) = \vec{v}$. Note that if $\sigma_{\vec{u}}^G = \tau_i$ where $l(\vec{u}) = j$, then the map across dimension $j+1$ within the subcube of dimension $k-j$ that fixes the first j bits of $\vec{u}$ toggles both bits $j+1$ and $j+1+i$.

Example. *T_1, a twisted hypercube of dimension 3 and radius 2.* The twisted hypercube T_1, as shown in Figure 7.1 is defined by using the identity for all associated permutations except that $\sigma_1^{T_1}(\vec{v}) = \tau_1(\vec{v})$. The matching among vertices with first coordinate 1 thus toggles both the second and third bits. Note that the subscript on T_n is the logarithm of its radius. We will construct a sequence of twisted hypercubes with exponentially growing radius but even faster-growing dimension. □

7.1. The construction

Given a twisted hypercube $H \in \mathcal{G}_k$ with radius r, with $\vec{0}$ as the center vertex), we construct a twisted hypercube $G \in \mathcal{G}_{2k+r}$ that we will show has radius at most $2r$, again with $\vec{0}$ as the center vertex. Since vertices in G have length $2k+r$, we must specify $\sigma_{\vec{u}}^G(\vec{v})$ for each $\vec{u}, \vec{v}$ with $l(\vec{u}) + l(\vec{v}) = 2k + r - 1$.

When $0 \leq l(\vec{u}) < k$, partition $\vec{v}$ by letting the first $k - 1 - l(\vec{u})$ bits be $\vec{c}$ and the last $k + r$ bits be $\vec{d}$. Define $\sigma_{\vec{u}}^G(\vec{v}) = \sigma_{\vec{u}}^H(\vec{c}) : \vec{d}$. For $0 \leq j < k$ this makes $f_{G,j}(\vec{v})$ act like $f_{H,j}$ on the first k bits of $\vec{v}$ and leave the rest unchanged.

When $k \leq l(\vec{u}) < 2k$, partition $\vec{u}$ by letting the first k bits be $\vec{a}$ and the last $l(\vec{u}) - k$

bits be $\vec{b}$. Also partition $\vec{v}$ by letting the first $k-1-l(\vec{b})$ bits be $\vec{c}$ and the last r bits be $\vec{d}$. Define $\sigma^G_{\vec{u}}(\vec{v}) = \sigma^H_{\vec{b}}(\vec{c}) : \tau_{d_H(0,\vec{a})}(\vec{d})$. For $k \leq j < 2k$ this makes $f_{G,j}$ leave the first k bits of $\vec{v}$ unchanged, while it acts like $f_{H,j-k}$ on the next k bits, and (possibly) toggles one bit from the last r. The bit toggled depends on how far $\vec{a}$ is from $\vec{0}$ in H.

When $2k \leq l(\vec{u}) < 2k+r$, define $\sigma^G_{\vec{u}}$ be the identity permutation. For $2k \leq j < 2k+r$ this makes $f_{G,j}$ change only the jth bit, that is, it acts like $f_{Q_r,j-2k}$ on the last r bits, while leaving all others unchanged.

Example *T_1 is the twisted hypercube obtained from $T_0 = K_1$ by this construction.* The only twisted hypercube of dimension 1 is K_1, its only associated permutation is the identity. Thus, only the second rule above results in an associated permutation for T_1 that is not the identity. That rule says that $\sigma^{T_1}_1(\vec{v}) = \tau_1(\vec{v})$, while $\sigma^{T_1}_0(\vec{v}) = \tau_0(\vec{v})$, which is the identity. This is precisely our previous definition of T_1. □

Proposition 7.1.1. *If H has radius r, then the radius of G obtained from H by the above construction is at most $2r$.*

Proof: We present an algorithm to travel from a given vertex $\vec{v}$ to $\vec{0}$ in no more than $2r$ steps.

Write $\vec{v} = \vec{w}_1 : \vec{w}_2 : \vec{w}_3$ where $l(\vec{w}_1) = l(\vec{w}_2) = k$ and $l(\vec{w}_3) = r$. To begin, set $\vec{a} = \vec{w}_1, \vec{b} = \vec{w}_2$, and $\vec{c} = \vec{w}_3$. While constructing a path to the origin, we let $\vec{a} : \vec{b} : \vec{c}$ represent the current vertex.

By the notes about $f_{G,j}$ above, there are three types of operations we can perform. There are k incident edges of the first type, k of the second type, and r of the third type. We can 1) treat $\vec{a}$ as a vertex of H, 2) treat $\vec{b}$ as a vertex of H while toggling the $d_H(\vec{0}, \vec{a})$th bit of $\vec{c}$, or 3) toggle one bit in $\vec{c}$. By carefully choosing the order in which these three types of operations are repeated, we will reach $\vec{a} = \vec{b} = \vec{c} = \vec{0}$ in no more than $2r$ steps. The general strategy is to take no more than r steps to reach $\vec{a} = 0$, interspersed with no more than r steps to reach $\vec{b} = 0$. Since steps of the second type also affect $\vec{c}$, we order the steps so that we zero out bits of $\vec{c}$ for free. Only if we need fewer than r steps of the first or second type will we ever take steps of the third type.

While $d_H(\vec{0}, \vec{a}) > 0$, the algorithm repeats the following: if $c_{d_H(\vec{0},\vec{a})} = 1$ it toggles $c_{d_H(\vec{0},\vec{a})}$, using the operation of type 2 that moves $\vec{b}$ closer to $\vec{0}$ in H if $d_H(\vec{0}, \vec{b}) > 0$ and using $f_{G,2k+d_H(\vec{0},\vec{a})}$ (which is of type 3) otherwise. (Note that all type 2 operations on $\vec{a} : \vec{b} : \vec{c}$ toggle the same bit in $\vec{c}$.) It then uses the operation of type 1 that moves $\vec{a}$ one step closer to $\vec{0}$ in H. If $c_{d_H(\vec{0},\vec{a})} = 0$ it does only the second of these operations.

This part of the algorithm repeats $d_H(\vec{0}, \vec{w}_1)$ times, using either one or two steps per iteration. All together, it uses $d_H(\vec{0}, \vec{w}_1) + \sum_{i=1}^{d_H(\vec{0},\vec{w}_1)}$ steps. When it terminates, $\vec{a} = \vec{0}$. Also $d_H(\vec{0}, \vec{b}) \leq r - \sum_{i=1}^{d_H(\vec{0},\vec{w}_1)}$, and no more than $r - d_H(\vec{0}, \vec{w}_1)$ bits of $\vec{c}$ are equal to 1 (since for each $c_i = 1$ with $i \leq d_H(\vec{0}, \vec{w}_1)$, c_i has been toggled). Thus we need at most $r - \sum_{i=1}^{d_H(\vec{0},\vec{w}_1)}$ operations of type 2 (treating $\vec{b}$ as a vertex of H) to bring $\vec{b}$ to $\vec{0}$. We can additionally use the operation $f_{G,2k+i}$ of type 3 to toggle each c_i where $c_i = 1$. As noted, there are at most $r - d_H(\vec{0}, \vec{w}_1)$ such i.

The total number of steps used is thus at most $2r$. □

Example. *A path of length* 4 *from* 11111111 *to* 00000000 *in* T_2*, the twisted hypercube obtained from* T_1 *by the above construction.* The three types of operations discussed above do the following. Type 1 treats the first three bits as a vertex of T_1, leaving the rest unchanged. Type 2 treats the next three bits as a vertex of T_1, while (possibly) toggling one of the last two bits. The bit toggled depends on how far from $\vec{0}$ the first three bits are, considered as a vertex of T_1. Type 3 toggles either of the last two bits. Thus we can proceed from 11111111 to 11110010 by an operation of the second type, moving the second three bits from 111 to 100 in T_1 while toggling the second bit (of the last two bits) because the first three bits are 111, which has distance 2 from $\vec{0}$. We then treat the first three bits as a vertex of T_1 and proceed from 11110010 to 10010010. Then a step similar to the first gives us 10000000, and we finish by taking the last step to 00000000, again treating the first three bits as a vertex of T_1. □

Now let H_0 be a twisted hypercube of dimension a and radius b. For $i \geq 1$, let H_i be the twisted hypercube obtained by performing the construction on H_{i-1}. The radius r_i of H_i is $2^i b$. The dimension k_i of H_i is $2k_{i-1} + r_{i-1}$. We must solve the recurrence $k_0 = a$, $k_i = 2k_{i-1} + r_{i-1}$ to find a general form for the dimension. A simple inductive proof shows that the solution is $k_i = 2^i a + i2^{i-1}b = 2^i(a + \frac{ib}{2})$. Specifically, if we perform this construction with $T_0 = K_2$ which has radius $b = r_0 = 1$ and dimension $a = 1$, then we have $k_i = 2^{i-1}(i+2)$ and $r_i = 2^i$.

Theorem 7.1.2. *The minimum diameter among twisted hypercubes of dimension k is in $\Theta(k/\lg k)$.*

Proof: Let $k_i = 2^{i-1}(i+2)$ be the dimension of T_i. Given k, choose i so that $k_{i-1} \leq k \leq k_i$. Let H be the subgraph of T_i induced by the vertices whose leftmost $k_i - k$ bits are 0. Observe that H has radius at most $r(T_i)$ since the algorithm avobe does not leave H when traveling from a given vertex of H to $\vec{0}$. This means that, for $k \leq k_i$, the minimum radius of a k-dimensional twisted hypercube is no more than $r(T_i)$. Because the last factor has limit 1, we have

$$2^i \leq 2^i(1 + \frac{\epsilon}{4})\frac{i+1}{i - 1 + \lg(i+2)}$$

for any ϵ and sufficiently large i. Rewriting, we have

$$r(H) \leq r(T_i) = 2^i \leq (4+\epsilon)\frac{2^{i-2}(i+1)}{\lg(2^{i-1}(i+2))} = \frac{k_{i-1}}{\lg k_i} \leq \frac{k}{\lg k}$$

where the last step uses the choice of i so that $k_{i-1} \leq k \leq k_i$.

Since the diameter of a graph is no more than twice the radius, this proves, for sufficiently large k, that the minimum diameter of a k-dimensional twisted hypercube is bounded above by a constant times $\frac{k}{\lg k}$. We may choose the constant to be any number larger than 8. Since we have already shown that the diameter of a k-dimensional twisted hypercube is at least $\frac{k}{\lg k} - 1$, this completes the proof. □

Another parameter of interest when considering twisted hypercubes and their diameter is the routing runtime, loosely defined as the time it takes a vertex v to decide which neighbor is the first step of the shortest path to another vertex w. Here we note only that, for the problem of traveling from v to w, we have only shown how to travel from v to $\vec{0}$ to w. The first part of this path, from v to $\vec{0}$ can have constant routing runtime; we can calculate in advance the first step from v to $\vec{0}$ for each vertex v and store that at v. To travel from 0 to w, however, each vertex along the path must run the above algorithm in reverse, finding the path from w to $\vec{0}$, so that it knows the next step. If each processor can store the first step for every vertex (a possibly unreasonable requirement), then the routing time would be linear in the distance from $\vec{0}$ to the destination.

8. Domination number of Kneser graphs

In this chapter we study dominating sets in a family of graphs of classical interest. A *dominating set* S of a graph G is a a set $S \subseteq V(G)$ such that every vertex of $V(G) \setminus S$ is adjacent to some vertex in S. A *total dominating* set S of a graph G is a set $S \subseteq V(G)$ such that every vertex of $V(G)$ is adjacent to some vertex in S. The *domination number* $\gamma(G)$ of a graph G is the minimum size of a dominating set, and the *total domination number* $\gamma_t(G)$ is the minimum size of a total dominating set. Always $\gamma(G) \leq \gamma_t(G)$, since a total dominating set is also a dominating set.

The Kneser graph $K(n,k)$ has as vertices the k-sets of $[n]$. Two vertices of $K(n,k)$ are adjacent if the k-sets are disjoint. In this context, we often call the elements of $[n]$ *points*. When $n < 2k$ the Kneser graph is an independent set; when $n = 2k$ it is a matching. Thus, we consider only $n \geq 2k+1$. Let $\gamma(n,k) = \gamma(K(n,k))$, and let $\gamma_t(n,k) = \gamma_t(K(n,k))$.

As mentioned in Chapter 1, the value of this parameters on the Kneser graph is of interest to graph theorists and has applications to design theory.

The problems of determining the domination number and total domination number of a Kneser graph can be restated in terms of blocking sets. A *blocking set* for a collection S of subsets of $[n]$ is a set $B \subseteq [n]$ such that B intersects every set in S. A collection S of k-sets of $[n]$ is a total dominating set of $K(n,k)$ if and only there is no k-element blocking set for S. Such a collection S is a dominating set of $K(n,k)$ if and only if every k-element blocking set for S belongs to S.

It is easy to show that $\gamma_t(n,k) \leq k+1$ when $n \geq k^2+k$. Every collection of $k+1$ pairwise disjoint k-sets is a total dominating set, since no k-set can intersect all $k+1$ of these sets. Clark [6] showed that also $\gamma(n,k) \geq k+1$ when $n \geq k^2+k$. He further showed that $\gamma_t(n,k) = \gamma(n,k) = k+2$ when $k^2 + k - \frac{k}{2} \leq n < k^2 + k$. We extend these results to show that $\gamma(n,k) = \gamma_t(n,k) = k+t+1$ when $k^2+k-t\lfloor\frac{k}{2}\rfloor \leq n < k^2+k-(t-1)\lfloor\frac{k}{2}\rceil$ and $0 \leq t \leq \lceil\frac{k}{2}\rceil$. We prove that $\gamma_t(n,k) \leq k+t+1$ (by construction), and we prove that $\gamma(n,k) \geq k+t+1$ for. Since the bounds are equal and $\gamma(G) \leq \gamma_t(G)$, this implies that $\gamma(n,k) = \gamma_t(n,k)$ in the given range.

Clark also noted that $\gamma_t(n,k)$ is nonincreasing in n, that is, that $n' \geq n$ implies $\gamma_t(n',k) \leq \gamma_t(n,k)$, and he asked whether the corresponding statement holds for $\gamma(k,n)$. We prove that it does.

8.1. Monotonicity of $\gamma(n,k)$

We first prove that $\gamma_t(n,k)$ is nonincreasing in n.

Theorem 8.1.1. *If $n' \geq n$, then $\gamma_t(n',k) \leq \gamma_t(n,k)$.*

Proof: We show that every total dominating set S of $K(n,k)$ is also a total dominating set of $K(n',k)$. For every k-set A of $[n']$, $A \cap [n]$ has no more than k elements. Let C be any k-set of $[n]$ containing $A \cap [n]$. Since S is a total dominating set in $K(n,k)$, some $S_i \in S$ is disjoint from C. Since $S_i \cap A \subseteq S_i \cap C$, also $S_i \cap A = \emptyset$. Thus, every k-set A is disjoint from some set in S, and S is a total dominating set in $K(n',k)$. □

We cannot use the same argument to prove the monotonicity of $\gamma(n,k)$. It could happen that every extension C of $A \cup [n]$ for a given set $A \subseteq [n']$ belongs to S. In this case, S would be a dominating set in $K(n,k)$ without being a dominating set in $K(n',k)$. We prove that this cannot happen when S is a minimal dominating set in $K(n,k)$ and $n' = n+1$.

Theorem 8.1.2. *If $n' \geq n \geq 2k+1$ then $\gamma(n',k) \leq \gamma(n,k)$.*

Proof: It suffices to prove only that $\gamma(n+1,k) \leq \gamma(n,k)$. Let S be a minimal dominating set in $K(n,k)$. We show that S is a dominating set in $K(n+1,k)$. Let A be an arbitrary k-set in $K(n+1,k)$. We must find a set $S_i \in S$ so that $S \cap A = \emptyset$. If $A \subseteq [n]$, then such S_i exists because S is a dominating set in $K(n,k)$. If, on the other hand, $n+1 \in A$, let $C = \{C_i : C_i \text{ is a } k\text{-set of } [n] \text{ and } (A \cap [n]) \subseteq C_i\}$ be the set of k-sets of $[n]$ extending $A - \{n+1\}$. Note that $|C| = n-k+1$ since there are $n-(k-1)$ elements in $[n] \setminus A$. If some $C_i \in C$ is not in S, then the argument of Theorem 8.1.1 applies. Thus we may assume that $C \subseteq S$. For each $C_i \in C$, let c_i denote the point in C_i but not in A. We claim that each $S - C_i$ is a dominating set in $K(n,k)$. If not, then for some i there is a k-set D such that D is disjoint from C_i but D intersects every C_j with $j \neq i$. Thus $c_j \in D$ for all $j \neq i$. This implies that D must have $n-k$ elements, since the c_j's are distinct. Since $n-k \geq k+1$, we have obtained a contradiction. □

8.2. The upper bound

We prove the upper bound for $\gamma_t(n,k)$ by construction, finding a total dominating set of the required size.

Theorem 8.2.1. *$\gamma_t(n,k) \leq k+t+1$ for $n \geq k^2+k-t\lfloor \frac{k}{2} \rfloor$, where $t \leq \lceil \frac{k}{2} \rceil$.*

Proof: We prove the claim only for $n = k^2+k-t\lfloor \frac{k}{2} \rfloor$. Theorem 8.1.1 then implies that it holds for all larger n. We use the observation that a collection S of k-sets is a total dominating set in $K(n,k)$ if every set intersecting all its members has more than k elements. We construct such a collection S of size $k+t+1$.

When k is even, we write n as $k^2-(t-2)\frac{k}{2}$. We use two configurations, A and T, as depicted in Figure 8.1; A is simply a k-set. The configuration T consists of three k-sets,

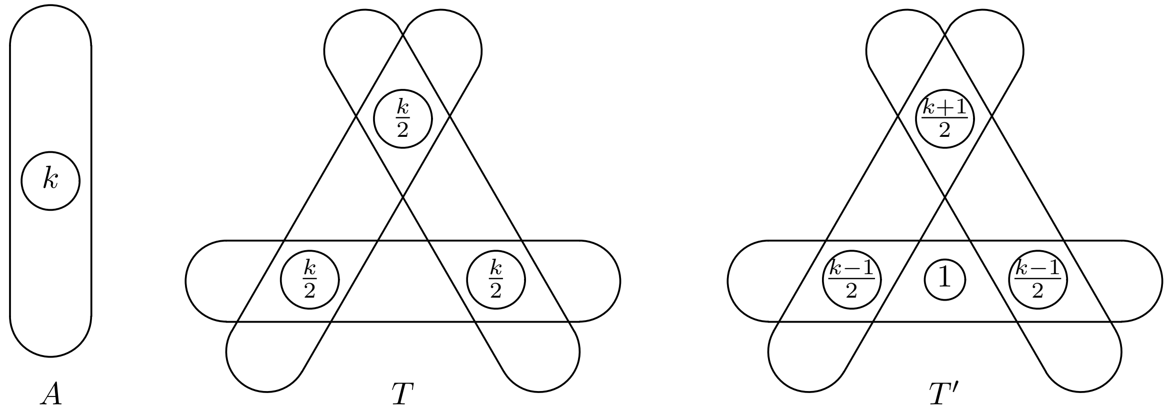

Figure 8.1.

corresponding to three groups of $\frac{k}{2}$ points taken two groups at a time. Note that it takes at least two points to intersect all three k-sets of T. Let S, a collection of k-sets, consist of $k-2t+1$ copies of A and t copies of T, all pairwise disjoint. Then S is a collection of $(k-2t+1)+3t$ k-sets whose union has $k(k-2t+1)+t(3\frac{k}{2}) = k^2-(t-2)\frac{k}{2} = n$ points. Suppose that some set B intersects every member of S. Then B must have at least one element from each copy of A, and at least two elements from each copy of T. Since the copies are pairwise disjoint, B must have $(k-2t+1)+2t = k+1$ elements. Thus, no k-set intersects all members of S, and S is a total dominating set in $K(n,k)$.

When k is odd, we write n as $k^2-(t-2)\frac{k}{2}+\frac{t}{2}$. The construction is similar to the even case, but we must use a different configuration T' instead of T since $\frac{k}{2}$ is no longer an integer (see Figure 8.1). Like T, the configuration T' consists of three k-sets, but here we begin with one group of $\frac{k+1}{2}$ points, two groups of $\frac{k-1}{2}$ points, and one extra point. The three k-sets are formed by taking the groups two at a time, the extra point is used to make the set formed by the two groups of $\frac{k-1}{2}$ points have k points in it. Note that T' uses a total of $\frac{3k+1}{2}$ points. As with T, it takes at least two points to interesect the three

k-sets of T'. Let S consist of $k-2t+1$ copies of A and t copies of T', all disjoint. Then S is a set of $(k-2t+1)+3t$ k-sets on $k(k-2t+1)+t(\frac{3k+1}{2}) = k^2-(t-2)\frac{k}{2}+\frac{t}{2} = n$ points. Again, a set B intersecting every member of S must have at least $(k-2t+1)+2t = k+1$ elements, and thus S is a dominating set. $\square$

8.3. The lower bound

We begin with some necessary lemmas.

Lemma 8.3.1. *Every multigraph with maximum degree k and e edges contains a matching of size $\lceil \frac{e}{\lfloor 3k/2 \rfloor} \rceil$.*

Proof: Shannon's Theorem [27] states that a multigraph with maximum degree k is $\lfloor \frac{3k}{2} \rfloor$-edge-colorable. In such a coloring, each color is used on an average of $\frac{e}{\lfloor 3k/2 \rfloor}$ edges. Some color must therefore be used on at least $\lceil \frac{e}{\lfloor 3k/2 \rfloor} \rceil$ edges. These edges are pairwise disjoint, and thus they form the required matching. $\square$

Lemma 8.3.2. *Let S be a set of $k+t$ k-sets from a set $[n]$ such that $n \geq 2k+t$. If S has a blocking set of size $k-1$, then S is not a dominating set of $K(n,k)$.*

Proof: A blocking set of size $k-1$ is extendable to a blocking set of size k in $n-(k-1) \geq k+t+1$ ways. Only $k+t$ of these can be in S. Therefore, there is a blocking set of size k that is not in S. This set is a vertex of $K(n,k)$ that is not in S and is not adjacent to any vertex in S. $\square$

We now prove a technical lemma of independent interest.

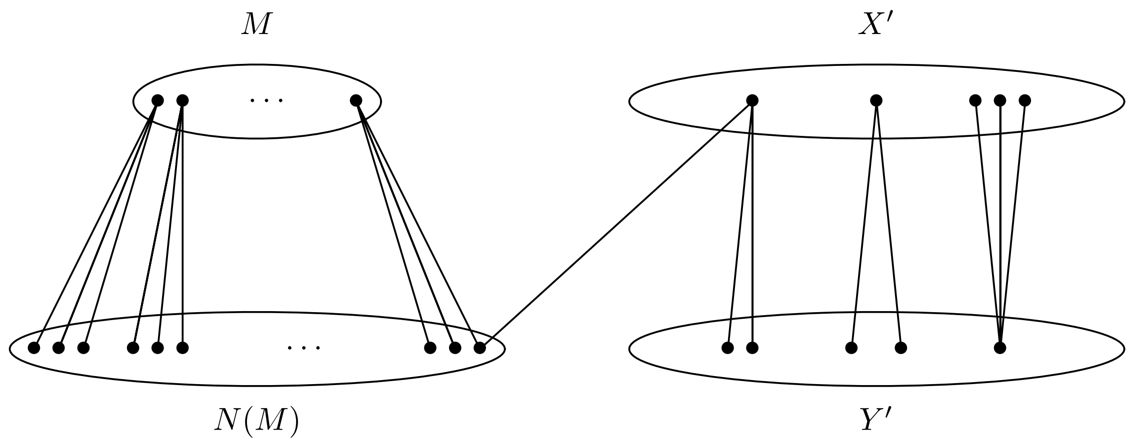

Figure 8.2

Lemma 8.3.3. *If $n < k^2 + k - (t-1)\lfloor \frac{k}{2} \rfloor$ and S is a collection of $k+t$ sets of size k in $[n]$, then $[n]$ contains a set J that intersects $|J| + t$ members of S.*

Proof: Let G be the incidence graph between points in $[n]$ and members of S. That is, G is the bipartite graph with partite sets $X = [n]$ and $Y = S$ that has an edge from $x \in X$ to $S_y \in Y$ if and only if $x \in S_y$. We claim that X contains a set J such that $|N(J)| = |J| + t$; in other words, J intersects at least $|J| + t$ sets in S. We begin by iteratively selecting a set $M \subseteq X$. As long as such a vertex is available, we select a vertex for M that has at least three neighbors not yet adjacent to any vertex of M. We stop when no such vertex remains. If m is the number of points selected for M, then we have $|N(M)| = m + l$ with $l \geq 2m$. Furthermore, with $X' = X \setminus M$ and $Y' = Y \setminus N(M)$, each element of X' has at most two neighbors in Y' (see Figure 8.2).

Now let G' be the subgraph of G induced by X' and Y'. Each vertex in Y' corresponds to a k-set, and none of these k-sets contain points in M since $Y' = Y \setminus N(M)$. Thus, each of the $(k + t - m - l)$ vertices in Y' is adjacent to k points in X', so G' has exactly $k(k + t - m - l)$ edges. Since there are $k(k + t - m - l)$ edges to the $n - m$ points in X', there must be at least $k(k+t-m-l)-(n-m)$ points in X' that have degree 2 in G'.

Now form a multigraph H on the $k+t-m-l$ sets of S that remain in Y'. Place an edge between two sets for each point in X' that is in both of them. That is, H has an edge for each vertex in X' with degree 2 in G'. This yields a multigraph H with at least $k(k+t-m-l)-(n-m)$ edges. We claim that H has a matching with at least $t-l$ edges. By Lemma 8.3.1, it suffices to show that H has more than $(t-l-1)\lfloor\frac{3k}{2}\rfloor$ edges. Since $n \le k^2+k-(t-1)\lfloor\frac{k}{2}\rfloor-1$, we have

$$\begin{aligned} e(H) &\ge k(k+t-m-l)-(k^2+k-(t-1)\lfloor\frac{k}{2}\rfloor-1-m) \\ &= (t-1)\lfloor\frac{k}{2}\rfloor+k(t-m-l-1)+m+1 \\ &= (t-l-1)\lfloor\frac{3k}{2}\rfloor+l\lfloor\frac{k}{2}\rfloor-(k-1)m+1 \end{aligned}$$

Since $l \ge 2m$, we have $l\lfloor\frac{k}{2}\rfloor \ge 2m(k-1)/2=(k-1)m$, and the desired bound holds.

Now, since each edge in H corresponds to a point in X' that is in two sets in Y', a matching of size $t-l$ in H corresponds to a set of $t-l$ points that intersects $2t-2l$ sets in Y'. Together with M, this forms our set J of $j=m+t-l$ points with $|N(J)| = (m+l)+(2t-2l)=m+2t-l=j+t$. □

We now prove the lower bound.

Theorem 8.3.4. *$\gamma(n,k) \ge k+t+1$ if $n < k^2+k-(t-1)\lfloor\frac{k}{2}\rfloor$, where $0 \le t \le \lceil\frac{k}{2}\rceil$.*

Proof: Let S be a collection of $k+t$ k-sets for some n, k, and t fulfilling the hypotheses. We prove that S is not dominating by finding a blocking k-set B for S such that B is not in S. By Lemma 8.3.3, we have a set $J \subseteq [n]$ that intersects $|J|+t$ members of S.

Since each member of S has size k, we can now choose $k-|J|$ additional distinct points, one from each of the remaining $k+t-(|J|+t)$ sets in S, to form a k-set B that intersects

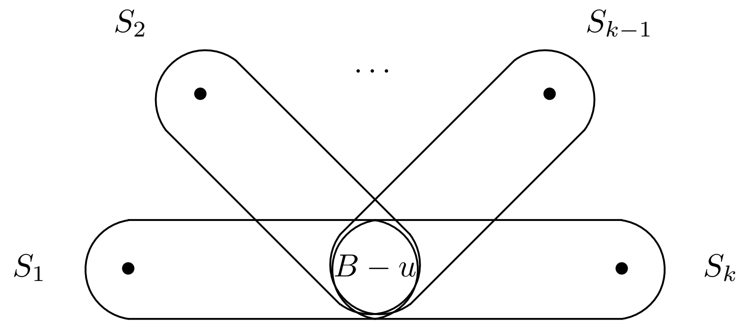

Figure 8.3

every set in S.

We claim there must be some point u of B such that $B-u$ intersects all sets in S except for at most one. If not, then for each $b \in B$ there would be at least two sets in S whose only intersection with B would be b. These must be distinct, but S has only $k+t < 2k$ sets.

If $B-u$ intersects all sets of S, then Lemma 8.3.2 completes the proof. Otherwise, let U be the only set of S not intersected by $B-u$. Thus $U \cap B = \{u\}$. There are k ways to augment $B-u$ using an element of U. If any of these k-sets is not in S we have found a blocking k-set for S that is not in S, which corresponds to a vertex of $K(n,k)$ that is not dominated. If each of these k-sets is in S, then $B-u$ is a common $(k-1)$-set for k elements of S (see Figure 8.3). In this case, S has a blocking set of size no greater than $t+1$, consisting of one point from $B-u$ and at most one point from each of the remaining t other sets in S. Since $t+1 < k-1$, Lemma 8.3.2 implies that S is not a dominating set.

□

8.4. Constructions for smaller n.

In this section we describe the best construction we have found for the full spectrum of

n. It generalizes the construction of Section 8.2 and may be optimal, at least near the top of the range.

We can form a configuration of k-sets generalizing A and T from Section 8.2 by starting with a groups of $\frac{k}{b}$ points and taking all sets of b groups as k-sets. With $a = b = 1$ we get configuration A, and with $a = 3$ and $b = 2$ we get configuration T. This a, b-configuration consists of $\binom{a}{b}$ k-sets and uses a total of $\frac{ak}{b}$ points. It takes a set of size at least $a - b + 1$ to intersect all $\binom{a}{b}$ k-sets, because $a - b$ points intersect at most $a - b$ groups, leaving b groups which form a k-set that has not been intersected.

Form a collection S of k-sets by taking $\lceil \frac{k+1}{a-b+1} \rceil$ pairwise disjoint copies of the a, b-configuration. This collection S uses $\lceil \frac{k+1}{a-b+1} \rceil \frac{ak}{b}$ points and forms $\lceil \frac{k+1}{a-b+1} \rceil \binom{a}{b}$ k-sets. It requires at least $(a-b+1)\lceil \frac{k+1}{a-b+1} \rceil \geq k+1$ points to intersect each k-set of S. This means S is a total dominating set in $K(n, k)$ for $n = \lceil \frac{k+1}{a-b+1} \rceil \frac{ak}{b}$. This proves $\gamma_t(\lceil \frac{k+1}{a-b+1} \rceil \frac{ak}{b}, k) \leq \lceil \frac{k+1}{a-b+1} \rceil \binom{a}{b}$.

As in Section 8.2, we can form a collection S of k-sets using two types of configurations, say a, b and a', b'. If we take α copies of the a, b-configuration, and α' copies of the a', b'-configuration where $\alpha(a-b+1) + \alpha'(a'-b'+1) \geq k+1$ then we have a dominating set of size $\alpha\binom{a}{b} + \alpha'\binom{a'}{b'}$ on $n = \alpha\frac{ak}{b} + \alpha'\frac{a'k}{b'}$ points. This gives us a way to interpolate between two configurations.

Consider the bound $\gamma_t(\lceil \frac{k+1}{a-b+1} \rceil \frac{ak}{b}, k) \leq \lceil \frac{k+1}{a-b+1} \rceil \binom{a}{b}$ that we get from the a, b-configuration. We can represent this bound as a point $p_{a,b} = (\lceil \frac{k+1}{a-b+1} \rceil \frac{ak}{b}, \lceil \frac{k+1}{a-b+1} \rceil \binom{a}{b})$ on a graph with axes n and γ. Since a dominating collection of sets is still dominating if we add more sets to it, we can prove the bound represented by any point to the right of $p_{a,b}$. By the monotonicity of $\gamma_t(n, k)$ we can thus prove the bound represented by any point above and to the right of $p_{a,b}$.

By using collections formed from two types of configurations, we can (roughly) interpolate between the points representing the bounds those configurations give us. This interpolation will not be exact, as there are divisibility conditions on k, but it still enables us to prove something close to the bound represented by any point in the convex hull of the points $p_{a,b}$ as we vary a and b. Examination of the points $p_{a,b}$ for small a and b leads us to believe that the one parameter family represented by the $a, \lceil \frac{a}{2} \rceil$-configurations are the best. Perhaps the interpolations between them give the actual value of $\gamma_t(n, k)$, not just an upper bound. Note that Theorems 1.2.1 and 1.3.4 prove this for the interpolation between the $1, 1$-configuration and the $3, 2$-configuration. So far, we have not been able to prove this for any additional configurations. It is probable that a hypergraph matching result extending Shannon's theorem will be needed, and such a result appears not to be known.

References

[1] Jin Akiyama, Hiroshi Era, Severino V. Gervacio, and Mamoru Watanabe. Path chromatic numbers of graphs. *J. Graph Theory*, 13(5):569–575, 1989.

[2] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[3] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[4] C. A. Baker. Extended Skolem sequences. *J. Combin. Des.*, 3(5):363–379, 1995.

[5] Ján Černý, Mirko Horňák, and Roman Soták. Observability of a graph. *Mathematica Slovaca*, 46(1):21–31, 1996.

[6] W. Edwin Clark. Domination numbers of Kneser graphs – or minimum families of k-sets with no blocking sets. Draft, 1995.

[7] Lenore Cowen, Wayne Goddard, and C. Esther Jesurum. Defective coloring revisited. *J. Graph Theory*, 24(3):205–219, 1997.

[8] G. A. Dirac. Map-colour theorems. *Canadian J. Math.*, 4:480–490, 1952.

[9] P. Erdős and H. Hanani. On a limit theorem in combinatorial analysis. *Publ. Math. Debrecen*, 10:10–13, 1963.

[10] P. Franklin. A six color problem. *J. Math. Phys.*, 13:363–369, 1934.

[11] Wayne Goddard. Acyclic colorings of planar graphs. *Discrete Mathematics*, 91:91–94, 1991.

[12] P. J. Heawood. Map-colour theorem. *Quart. J. Pure Appl. Math.)*, 24:332–338, 1890.

[13] M. S. Jacobson. The spot-chromatic number of $K_{2t} \square K_{2t}$. Presentation at workshop in Discrete Mathematics. Louisville, KY, June, 1997.

[14] Tommy R. Jensen and Bjarne Toft. *Graph coloring problems.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[15] M. Kneser. Aufgabe 300. *Jber. Deutsch. Math.-Verein.*, 58, 1955.

[16] Grzegorz Kubicki. The spot-chromatic number. Presentation at workshop in Discrete Mathematics. Louisville, KY, June, 1997.

[17] Shawn M. Larson and Paul Cull. Smaller diameter in hypercube-variant networks. Technical Report 95-20-2, Oregon State University, Sep 1995.

[18] C. C. Lindner and C. A. Rodger. Decomposition into cycles. II. Cycle systems. In *Contemporary design theory*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 325–369. Wiley, New York, 1992.

[19] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[20] Maryam Mirzakhani. A small non-4-choosable planar graph. *Bull. Inst. Combin. Appl.*, 17:15–18, 1996.

[21] Mirko Horňák. Observability. Email to D.B. West, 1997.

[22] Mirko Horňák and Roman Soták. Observability of complete multipartite graphs with equipotent parts. *Ars Combinatoria*, 41:289–301, 1995.

[23] Mirko Horňák and Roman Soták. Asymptotic behaviour of the observability of q_n. To be published in Discrete Mathematics, 1997.

[24] K. S. Poh. On the linear vertex-arboricity of a planar graph. *J. Graph Theory*, 14(1):73–75, 1990.

[25] Gerhard Ringel and J. W. T. Youngs. Solution of the Heawood map-coloring problem. *Proc. Nat. Acad. Sci. U.S.A.*, 60:438–445, 1968.

[26] Vojtěch Rödl. On a packing and covering problem. *European J. Combin.*, 6(1):69–78, 1985.

[27] Claude E. Shannon. A theorem on coloring the lines of a network. *J. Math. Physics*, 28:148–151, 1949.

[28] P. Slater. Twisted hypercubes. Personal communication, 1997.

[29] Carsten Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994.

[30] Margit Voigt. List colourings of planar graphs. *Discrete Math.*, 120(1-3):215–219, 1993.

[31] Margaret L. Weaver and Douglas B. West. Relaxed chromatic numbers of graphs. *Graphs Combin.*, 10(1):75–93, 1994.

Vita

Chris Hartman was born in 1969 in Fairbanks, Alaska. He graduated with a B.S. in Computer Science and in Mathematics in 1991 from the University of Alaska, Fairbanks. He entered the University of Illinois at Urbana–Champaign in August of 1991, and worked as a teaching/research assistant until August of 1997. He and his wife Heather have a daughter named Sarah.

EXTREMAL PROBLEMS IN GRAPH THEORY

Christopher M. Hartman, Ph.D.
Department of Mathematics
University of Illinois at Urbana-Champaign, 1997
D. West, Advisor

We consider generalized graph coloring and several other extremal problems in graph theory. In classical coloring theory, we color the vertices (resp. edges) of a graph requiring only that no two adjacent vertices (resp. incident edges) receive the same color. Here we consider both weakenings and strengthenings of those requirements. We also construct twisted hypercubes of small radius and find the domination number of the Kneser graph $K(n,k)$ when $n \geq \frac{3}{4}k^2 + k$ if k is even, and when $n \geq \frac{3}{4}k^2 - k - \frac{1}{4}$ when k is odd.

The *path chromatic number* $\chi_P(G)$ of a graph G is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of paths. We answer some questions of Weaver and West [31] by characterizing cartesian products of cycles with path chromatic number 2.

We show that if G is a toroidal graph, then for any non-contractible chordless cycle C of G, there is a 3-coloring of the vertices of G so that each color class except one induces a disjoint union of paths, while the third color class induces a disjoint union of paths and the cycle C.

The *path list chromatic number* of a graph, $\hat{\chi}_P(G)$, is the minimum k for which, given any assignment of lists of size k to each vertex, G can be colored by assigning each vertex a color from its list so that each color class induces a disjoint union of paths. We strengthen

the theorem of Poh [24] and Goddard [11] that $\chi_P(G) \le 3$ for each planar graph G by proving also that $\hat{\chi}_P(G) \le 3$.

The *observability* of a graph G is least number of colors in a proper edge-coloring of G such that the color sets at vertices of G (sets of colors of their incident edges) are pairwise distinct. We introduce a generalization of observability. A graph G has a *set-balanced k-edge-coloring* if the edges of G can be properly colored with k colors so that, for each degree, the color sets at vertices of that degree occur with multiplicities differing by at most one. We determine the values of k such that G has a set-balanced k-edge-coloring whenever G is a wheel, clique, path, cycle, or complete equipartite multipartite graph. We prove that certain 2-regular graphs with n vertices have observability achieving the trivial lower bound $\min\{j : \binom{j}{2} \ge n\}$. Horňák conjectured that this is always so.

The *spot-chromatic number* of a graph, $\chi_S(G)$, is the least number of colors with which the vertices of G can be colored so that each color class induces a disjoint union of cliques. We generalize a construction of Jacobson to show that $\chi_S(K_{mt} \square K_{nt}) \le \frac{mnt}{m+n} + 2\min(m,n)$ whenever $m+n$ divides t. The construction is nearly optimal.

Twisted hypercubes, generalizing the usual notion of hypercube, are defined recursively. Let $\mathcal{G}_0 = \{K_1\}$. For $k \ge 1$, the family $\mathcal{G}_k$ of *twisted hypercubes* of *dimension* k is the set of graphs constructible by adding a matching joining two graphs in $\mathcal{G}_{k-1}$. We construct a family of twisted hypercubes of small diameter. In particular, we prove that the order of growth of the minimum diameter among twisted hypercubes of dimension k is $\Theta(k/\lg k)$.

The domination number $\gamma(G)$ of a graph G is the minimum size of a set S such that every vertex of G is in S or is adjacent to some vertex in S. The Kneser graph $K(n,k)$ has as vertices the k-subsets of $[n]$. Two vertices of $K(n,k)$ are adjacent if the k-subsets are disjoint. We determine $\gamma(K(n,k))$ when $n \geq \frac{3}{4}k^2 \pm k$ depending on the parity of k.