Subcolorings and the subchromatic number of a graph

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Abstract

We consider the subchromatic number $\chi_S(G)$ of graph G, which is the minimum order of all partitions of $V(G)$ with the property that each class in the partition induces a disjoint union of cliques. Here we establish several bounds on subchromatic number. For example, we consider the maximum subchromatic number of all graphs of order n and in so doing answer a question posed in [20]. We also consider bounds on $\chi_S(G)$ when the size and genus of G are known. We also consider the parameter when applied to planar and outerplanar graphs.

It is known that the problem of determining whether $\chi_S(G) \leq k$ is NP-complete for all $k \geq 2$. We extend this by showing it is NP-complete for $k = 2$ even when restricted to the class of planar triangle-free graphs with maximum degree four. As a corollary we see that showing a planar triangle-free graph of maximum degree four has a 1-defective chromatic number of two is NP-complete, answering a question of [8]. We show that determining whether $\chi_S(G) \leq 3$ is NP-complete for planar graphs.

We consider the subchromatic number of cartesian products of complete graphs and show a correspondence with a natural covering of matrices.

We close by producing bounds on the subchromatic number in terms of chromatic number as well as the product of clique number with chromatic number. Sharpness for graphs with fixed clique size is discussed.

Key words: subchromatic number, generalized colouring
1991 MSC: 05C15

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1 Introduction

We consider only finite, simple, undirected graphs. For undefined terms and concepts the reader is referred to [6] and [14]. By c, c_1, c_2 etc. we shall denote positive constants. A *coloring* is a partition of the vertex set. In a *proper coloring* each partition induces the empty set; that is, each class induces a union of complete graphs of order one. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of sets needed to properly color G. We relax this somewhat. A *subcoloring* is a partition of the vertex set where each class induces the disjoint union of cliques. If k sets are used in the subcoloring, we say G is *k-subcolored*. The subchromatic number — the minimum number of sets needed in a subcoloring — seems to have been first presented in [25] and generalized by the same authors in [4]. It was named and studied in [2] and further studied in [18], there titled the spot-chromatic number. The *edge subchromatic number* is the minimum number of colors needed to color the edges of a graph so that each color class induces a disjoint union of cliques. This concept was studied in [9]. In [10] the *partite-chromatic number* is studied; this parameter equals the subchromatic number of the complement. We note that no disjoint union of cliques induces a path of length two. Further, a path of length two has a subchromatic number of two. Hence, we could equivalently define the subchromatic number to be the minimum number of sets needed to partition the vertex set so that no set induces a path of length two. This is closely related to the 2-chromatic number discussed in [5] and [21]. The *2-chromatic number* of a graph G, denoted $\chi^{(2)}(G)$, is the fewest number of colors needed to color the vertices so that no path of length two is monochromatic. In this definition, paths are not necessarily induced.

2 Bounds

A *cocoloring* of G is a partition of $V(G)$ where each set in the partition induces a complete or empty graph. Clearly, every cocoloring is also a subcoloring. In [25] graphs that have the property that every subcoloring is also a cocoloring are characterized. The *cochromatic number* $z(G)$ of G is the minimum number of sets needed to cocolor G. This parameter was studied in [12] and [16] among other places. The 1-*defective chromatic* number $\chi_1(G)$ of G is the fewest number of sets needed to partition $V(G)$ so that each set induces a graph of maximum degree at most one. We see that $\chi^{(2)}(G) = \chi_1(G)$ for all G. The *c-chromatic number* $c(G)$ of G is the minimum order of a partition of $V(G)$ where no part contains four vertices that induce a path. This parameter is discussed in [4,15]. By definition we have the following.

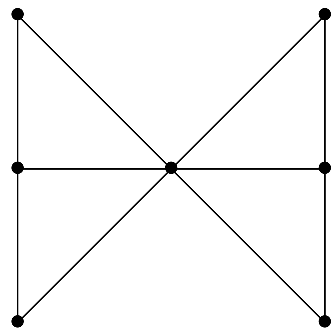

Fig. 1. An outerplanar graph with subchromatic number three

Remark 1 *For every graph G, the following inequalities hold.*

$$c(G) \leq \chi_S(G) \leq \min\{\chi(G), \chi(\overline{G}), z(G), \chi_1(G)\}$$

In [7] it is shown that every planar graph has 1-defective chromatic number at most four. The proof of this is independent of the Four Color Theorem. From this and Remark 1, we can establish the following without use of the Four Color Theorem.

Remark 2 *If G is a planar graph, then $\chi_S(G) \leq 4$.*

We will now discuss sharpness of this inequality.

Suppose G is a graph and v is a vertex not contained in G. Take two disjoint copies of G and join their vertices to v. If G has subchromatic number k, then this new graph has subchromatic number $k+1$. Thus, graphs of arbitrarily high subchromatic number exist. We can recursively construct a graph of order 2^{m-1} that has subchromatic number m. Suppose we begin with $G = K_1$ and perform this construction on G and then again on the resulting graph. We produce the graph in Figure 1. As every outerplanar graph can be 3-colored, we see the subchromatic number of every outerplanar graph is at most three and this is sharp. If we repeat this process once more, then we find a planar graph with subchromatic number four, thus showing Remark 2 is sharp.

Grötzsch's Theorem [17] tells us every planar triangle-free graph can be three colored. Thus, if G is planar and triangle-free, then G can be subcolored with three sets. As Figure 2 shows, this is best possible. Later, we shall see that it is non-trivial to prove that a given planar triangle-free graph has a subchromatic number equal to three.

We now consider complete multipartite graphs, and we demonstrate another method of constructing graphs with large subchromatic number.

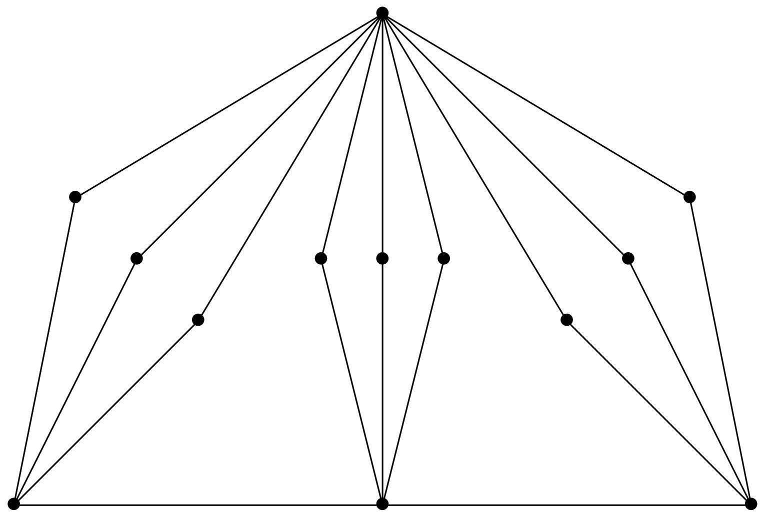

Fig. 2. A triangle-free planar graph with subchromatic number three

Remark 3 *With the preceding notation,* $\chi_S(K(1,2,\ldots,n)) = n$.

As noted in [25], a partition of a complete multipartite graph is a subcoloring if and only if it is a cocoloring. Hence, for complete multipartite graphs, the cochromatic and subchromatic numbers are the same. Remark 3 is thus a corollary of a result in [22], which gives the cochromatic number of any complete multipartite graph.

In [12] extremal problems are discussed for cochromatic number. Specifically, the existence is shown of constants c_1 and c_2, having the properties that if G has n vertices and m edges, then the cochromatic number of G is bounded above by both $c_1 n/\log n$ and $c_2\sqrt{m}/\log m$. Hence, the following:

Remark 4 *There exist constants* c_1 *and* c_2 *such that if* G *is a graph with order* n *and size* m*, then* $\chi_S(G) \leq c_1 n/\log n$ *and* $\chi_S(G) \leq c_2\sqrt{m}/\log m$.

We know [11] that there exists a constant c such that for each n, there is a graph of order n and size $\Theta(n^2)$ with the property that each subset of the vertex set having cardinality at least $c \log n$ contains three vertices that induce a path. Hence, both bounds in Remark 4 are best possible up to a constant coefficient. This answers, in asymptotic terms, a question posed in [20] (page 265).

A construction in [2] yields a graph G of order 13 with $\chi_S(G) = 4$. A result from [22] shows that if G has order eight, then $\chi_S(G) \leq 3$. In [2] the suspicion is stated that this can be "considerably improved." By Remark 3, this cannot be extended by more than one. In fact, after the following lemma we shall see that the most we can do is replace eight with nine.

Lemma 5 *If* G *is a graph of order six or seven, then one of the following* *holds:*

(1) $\chi_S(G) \leq 2$*,*
(2) G *contains a triangle,*
(3) G *contains an induced subgraph* H *of order four, with* $\chi_S(H) = 1$*.*

Strictly speaking, the third condition is superfluous. To prove this we could check all graphs of order seven or less (see [26,27]) or produce a longer proof. However, the statement of this lemma will serve our needs.

PROOF. Suppose G has order six or seven. If G contains no odd cycle, then it has chromatic number at most two. So, suppose G contains an odd cycle C. If C is a triangle, then we are done. So, suppose G contains no triangle. If C is a 5-cycle, then we may choose a vertex v not in C. If v is adjacent to three or more vertices of C, then G contains a triangle. So, v can be adjacent to at most two vertices in C. If v is adjacent to exactly two vertices, then these must be nonadjacent. So, we can select three vertices of C that induce $K_1 \cup K_2$ none of which are adjacent to v. These three vertices together with v induce a graph with subchromatic number one. So, suppose the smallest odd cycle in G has order seven. This cycle must be chordless, in which case G is a 7-cycle and thus has subchromatic number two. $\Box$

Theorem 6 *If* G *is a graph of order nine, then* $\chi_S(G) \leq 3$*.*

PROOF. We begin by showing that every graph of order five can be 2-subcolored. Clearly, the 5-cycle has subchromatic number two. If some graph has five vertices and is not the 5-cycle, then it must contain three vertices that induce a complete or empty graph. Giving these three vertices the same color and the remaining two a different color establishes the claim.

Suppose G is a graph of order nine. If G contains four vertices that induce a disjoint union of cliques, then place these four vertices in one class when subcoloring G. The remaining five vertices can be subcolored with two sets showing G has subchromatic number at most three. Specifically, if G contains a 4-clique or an empty graph on four vertices, then we are done.

From [2] (Corollary 17) we know that if the maximum degree of G is at most five, then $\chi_S(G) \leq 3$. So, suppose that G has a vertex of degree six or larger. If G contains a vertex v of degree eight, then let H be $G - v$. Now if H contains a triangle, then G has a 4-clique. So we may assume H contains no triangle nor empty graph of order four. There are exactly three graphs of order eight that do not contain a triangle nor empty graph of order four. They are listed in [3] (page 363). If we completely join any of these three graphs to a single vertex, then we create a graph with subchromatic number three. So we may assume the maximum degree of G is six or seven.

Suppose the maximum degree of G is seven. Let v be a vertex of G of degree seven. Let u be the vertex of G not adjacent to v. Let $H = G - \{u, v\}$. If H has subchromatic number two, then place $\{u, v\}$ in a separate class, giving G a 3-subcoloring. If H contains a triangle, then G contains a K_4. If H contains an induced subgraph of order four that has subchromatic number one, then $\chi_S(G)$ is at most three. By the preceding lemma, these cases are exhaustive and thus $\chi_S(G) \leq 3$.

The remaining case when the maximum degree is six follows in much the same fashion as seven and is omitted. $\square$

In [10] it is shown that the partite-chromatic number of a graph of order seven is at most three. As complementation does not change the order, the above discussion shows that this can be extended to graphs of order nine and that this is best possible.

For a natural number g, denote by $\chi_S(g)$ the maximum subchromatic number of all graphs of genus g. As we saw above, $\chi_S(0) = 4$. From [8] we know that if G is toroidal, then $\chi_1(G) \leq 5$. Hence, as conjectured in [2], $\chi_S(1) \leq 5$. It is unknown if this is sharp.

Remark 7 *With the preceding notation,* $\chi_S(g) = \Theta(\sqrt{g}/\log g)$.

PROOF. In [16] it was shown that if G has genus g, then $z(G) \leq c_1\sqrt{g}/\log g$. So, suppose G is a graph with at most $\sqrt{g}$ vertices. Such a graph has fewer than g edges, and hence has genus less than g. Replacing n in Remark 4 with $\sqrt{g}$ gives an expression asymptotic to the desired fraction. As the bounds in Remark 4 are sharp, the proof is complete. $\square$

3 Hardness

In this section, we consider the complexity of calculating the subchromatic number. As a corollary to work a result in [1] we know that computation of the subchromatic number is NP-complete. We present an alternative proof.

Theorem 8 *For any fixed $k \geq 2$, the problem of determining whether $\chi_S(G) \leq k$ is NP-complete.*

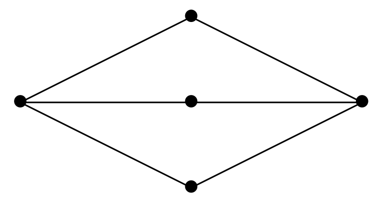

Fig. 3. A uniquely 2-subcolorable graph.

PROOF. Suppose $k \geq 3$ and G is a graph of order n. Form G' by taking $n+1$ copies of G and placing an edge between each pair of vertices that belong to distinct copies. We note that such a construction can be done in polynomial time. We further note that the subchromatic number of G' is equal to the minimum number of sets needed to partition $V(G)$ so that each set induces a clique. Hence, $\chi(G) = \chi_S\left(\overline{G}'\right)$. As the transformation from a graph to its complement can also be done in polynomial time, we see that the NP-complete problem of determining whether $\chi(G) \leq k$ reduces in polynomial time to the problem of determining whether $\chi_S(G) \leq k$ when $k \geq 3$. The case for $k = 2$ follows from Theorem 9 below. $\square$

Before proceeding to the next theorem, we shall develop a few gadgets [8,23]. Consider the graph in Figure 3. We note that this can be uniquely 2-subcolored. In such a subcoloring the vertices of degree two are placed in one set, and the other two vertices are placed in the second set. Using this idea, we see that the graph in Figure 4 can be uniquely 2-subcolored so that the two vertices labeled x are given the same color. We shall call this gadget an *extender* and denote it with the object shown at the top of Figure 5. We shall call the gadget shown in Figure 6 a *negater* and denote it with the object shown at the bottom of Figure 5. Note that a negator has a unique 2-subcoloring. In this coloring, the vertices labeled x and $\neg x$ must be given different colors. We note that if we take three extenders and identify a vertex of degree one in each, then we create a graph with a unique 2-subcoloring. In this coloring, each of the vertices of degree one must be given the same color. Extending this idea, we may create a planar, triangle-free graph of maximum degree four with an arbitrarily large number of vertices of degree one, all on the outer region, which must all be given the same color in any 2-subcoloring. Call such a gadget a *splitter* and refer to the vertices of degree one as *ports*. By a *conjugator* we mean the the gadget shown in Figure 7. We shall shortly note the conjugator's utility. Likewise, by an *uncrosser* we mean the gadget on the right in Figure 8. We note that in any 2-subcoloring of an uncrosser, the vertices labeled x must be given the same color, as must the vertices labeled y. We note that all gadgets constructed here are planar, triangle-free graphs with maximum degree four.

Albertson, Jamison, Hedetniemi and Locke [2] asked for a characterization of graphs with subchromatic number equal to two. They presented a number of

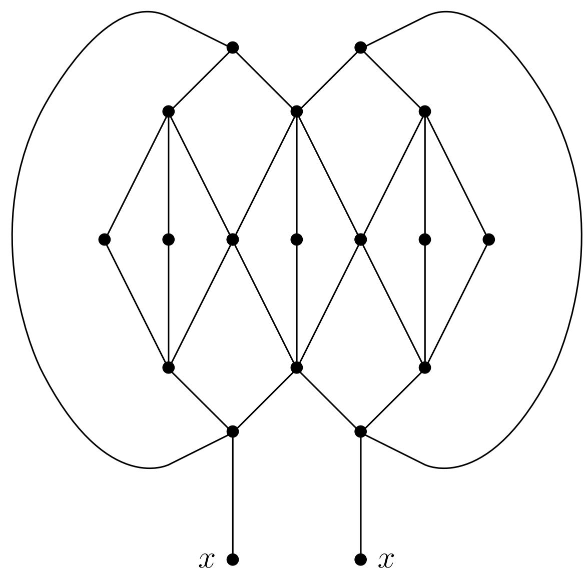

Fig. 4. The extender gadget.

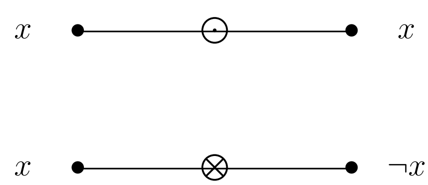

Fig. 5. Symbols for the extender and negator gadgets.

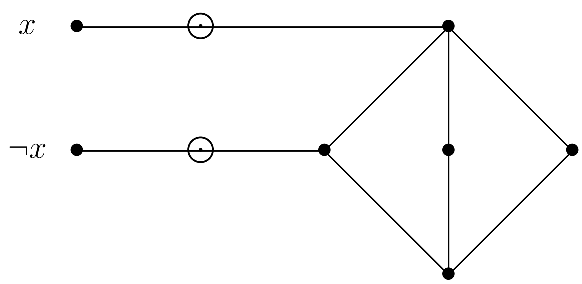

Fig. 6. The negator gadget.

interesting partial results that suggest that this is a non-trivial problem. The following adds further confirmation to this suggestion.

Theorem 9 *The problem of determining whether $\chi_S(G) \leq 2$ is NP-complete, even for planar triangle-free graphs of maximum degree four.*

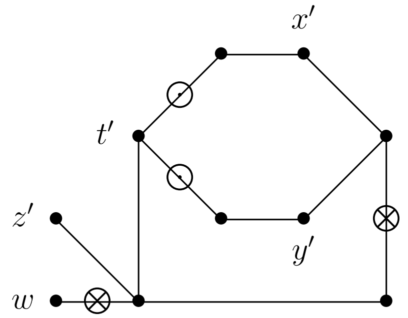

Fig. 7. The conjugator gadget.

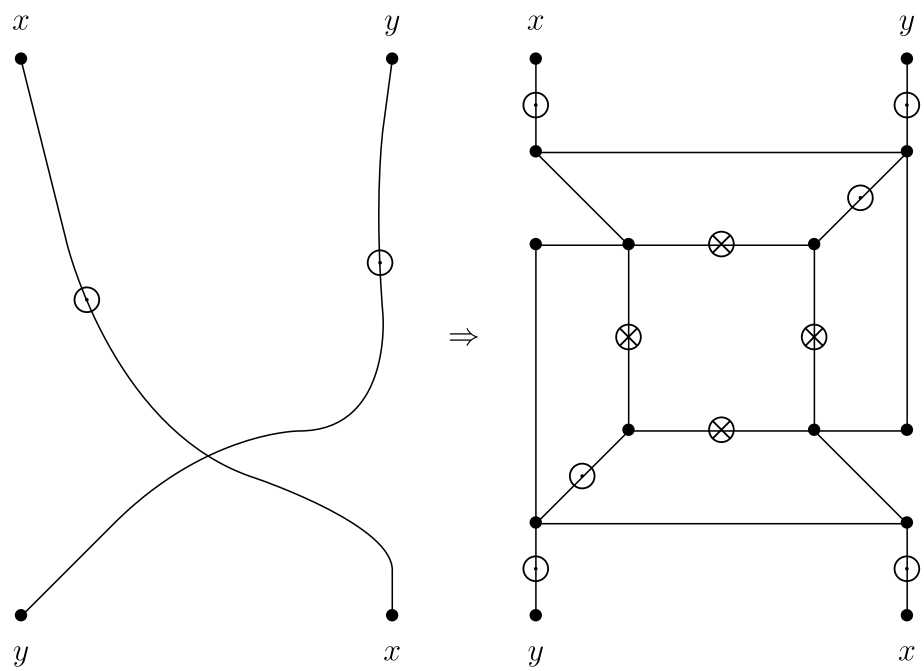

Fig. 8. Crossing extenders can be replaced with an uncrosser

PROOF. We proceed by reduction from the 3-satisfiability problem. Let $C = \{c_1, c_2, \ldots, c_n\}$ represent a set of clauses, where each clause consists of exactly three literals. We shall construct in polynomial time a corresponding planar, triangle-free graph G_C with maximum degree four, having the property that C is satisfiable if and only if G_C can be 2-subcolored. Let the literals used in C be denoted $x_1, x_2, \ldots, x_m$ and their negations $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m$. We begin building G_C with the vertices $x_1, x_2, \ldots, x_m, \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m$ and an additional vertex t. Now place between each x_j and $\overline{x}_j$ a negator. For each clause $x \vee y \vee z$ in C build a conjugator where the vertices x', y' and z' are attached by extenders to x, y and z respectively and t' and w are attached by extenders to vertex t. We note that the graph built at this point may be constructed in polynomial time. Further, the graph is triangle-free. It may be non-planar and may have maximum degree greater than four.

Let us now attempt to 2-subcolor this graph. If it is possible, label all vertices given the same color as t with 1. Label all other vertices 0. We associate true and false with 1 and 0, respectively. We note that vertex w in a conjugator built for the clause $x \vee y \vee z$ will be given the label corresponding to the truth value of $x \vee y \vee z$. As vertex w is joined by an extender to t, we see that a 2-subcoloring exists if and only if there is a truth assignment that satisfies all clauses in C.

It is likely that the graph constructed above has maximum degree greater than four and is non-planar. So, suppose v is a vertex of degree $m \geq 5$. Replace v with a splitter with m ports, and make each port adjacent to a neighbor of v. When this operation is performed on each vertex of degree greater than four we will reduce the maximum degree to four. In doing so, we will construct no triangles. Yet we still may not have a planar graph. However, we see that such a graph can be drawn in the plane with the only crossings being extenders crossing other extenders. Crossed extenders can be replaced with an uncrosser, as illustrated in Figure 8. In doing so, we create the graph G_C, which has no triangles, is planar and has maximum degree four. As building splitters and uncrossers may be done in polynomial time, and the number of uncrossers needed is $O(n^2)$, we see that G_C can be constructed in polynomial time. As C is satisfiable if and only if $\chi_S(G_C) \leq 2$, our proof is complete. $\square$

From the following we see that four cannot be replaced with three in the preceding theorem.

Remark 10 *If* $\Delta(G) \leq 3$*, then* $\chi_S(G) \leq 2$.

This follows as a special case of Corollary 17 in [2]. We present an alternate proof. We use a technique found in [24] and other places.

PROOF. Suppose G is a graph with maximum degree three. Color the vertices of G with two colors such that the number of monochromatic edges is minimized. If a vertex is adjacent to two others with the same color, then we could change its color and reduce the number of monochromatic edges. Hence, the vertices are partitioned into sets that induce graphs of maximum degree at most one. Since such a partition is a subcoloring, our proof is complete.

We note that in a triangle-free graph the subchromatic number and 1-defective chromatic number are the same. Hence the following, which answers a question posed in [8].

Corollary 11 *The problem of determining whether* $\chi_1(G) \leq 2$ *is NP-complete, even for triangle-free planar graphs of maximum degree four.*

Suppose G is a graph. Form G' in the following manner. For each vertex v of G, take a copy of the graph in Figure 1 and make each vertex in the copy adjacent to v. We note that G' and can be formed in polynomial time from G. Further, $\chi(G) \leq 3$ if and only if $\chi_S(G') \leq 3$. Also, G' is planar if and only if G is planar.

Remark 12 *The problem of determining whether $\chi_S(G) \leq 3$ is NP-complete for planar graphs.*

PROOF. This follows from the preceding discussion and the fact that by [13], the problem of determining whether $\chi(G) \leq 3$ is NP-complete for planar graphs. $\square$

4 Cartesian Products

In this section we investigate the subchromatic number of the cartesian product of cliques $K_m \,\square\, K_n$. When G is the cartesian product of two cliques K_m and K_n, this can be restated as a matrix labeling problem. Indeed, the matrix labeling problem is perhaps even more natural than the subcoloring problem. Represent $K_m \,\square\, K_n$ as an m by n matrix, where the entries of the matrix represent vertices. Two vertices are adjacent if they are in the same row or column. Call this the *corresponding* matrix.

Proposition 13 *A coloring of the entries of the corresponding matrix is a subcoloring of $K_m \,\square\, K_n$ if and only if no entry has another entry of the same color in both its row and its column.*

PROOF. Such a configuration of three entries corresponds to an induced monochromatic P_3, which we know is impossible. Conversely, if no such configuration exists, then entries with a common color occur in groups, all in one row, or all in one column. Since no entry appears in two such groups, the cliques corresponding to such groups are disjoint, and the color classes therefore induce disjoint unions of cliques. $\square$

It is easy to subcolor $K_n \,\square\, K_n$ with n colors; simply color each row with a different color. The greedy coloring algorithm, where we iteratively choose a color class as large as possible among the uncolored vertices, does no better. Jacobson [19] found a $(t/2+2)$-subcoloring of $K_t \,\square\, K_t$ when t is even, and proved this is optimal.

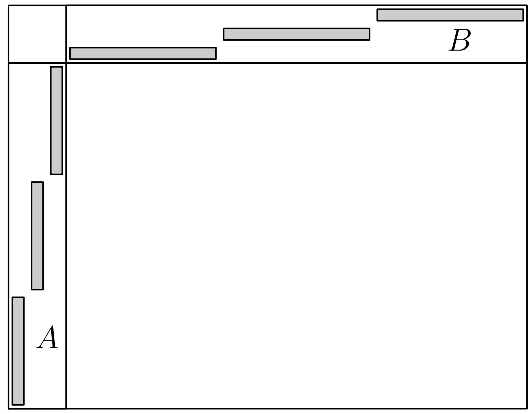

Fig. 9. The classes A and B from Proposition 14

Here we generalize Jacobson's coloring to show that

$$\chi_S(K_{mt} \square K_{nt}) \leq mnt/(m+n) + 2\min\{m,n\}$$

whenever $m+n$ divides t. This is nearly optimal. After our next proposition we will see that a subcoloring of $K_{mt} \square K_{nt}$ must use at least $mnt^2/(mt+nt-2)$ colors, and this is asymptotic to $mnt/(m+n)+2\min\{m,n\}$. First we give an upper bound on the size of a color class in a subcoloring of $K_m \square K_n$.

Proposition 14 *If m and n are at least two, then no color class in a subcoloring of $K_m \square K_n$ can contain more than $m+n-2$ vertices.*

PROOF. Consider a color class C. Partition C into two sets A and B by placing a vertex v into A if there is another vertex of C in v's column. Otherwise, put v into B. No vertex in A has another vertex of the same color in its row, by Observation 13 (see Figure 9). No vertex in B has another vertex of the same color in its column, by the choice of A. If A has vertices in c columns and B has vertices in r rows, then $|A| \leq m-r$ and $|B| \leq n-c$. If A or B is empty, then $|C| \leq \max\{m,n\}$, otherwise $|C| = |A|+|B| \leq m+n-(r+c) \leq m+n-2$, completing the proof. □

Jacobson's construction of a $(\frac{t}{2}+2)$-subcoloring of $K_t \square K_t$ when t is even is illustrated in Figure 10. There are two types of color classes: "stair-step", represented by black and shades of gray, and "diagonal", represented by the numbers 1 and 2. Since each element of such a stair-step class has neighbors of the same class only in its row or only in its column, each such class induces a disjoint union of cliques. Also, classes 1 and 2 are independent sets, that is, tK_1.

Theorem 15 *If $m+n$ divides t, then $\chi_S(K_{mt} \square K_{nt}) \leq \frac{mnt}{m+n} + 2\min\{m,n\}$.*

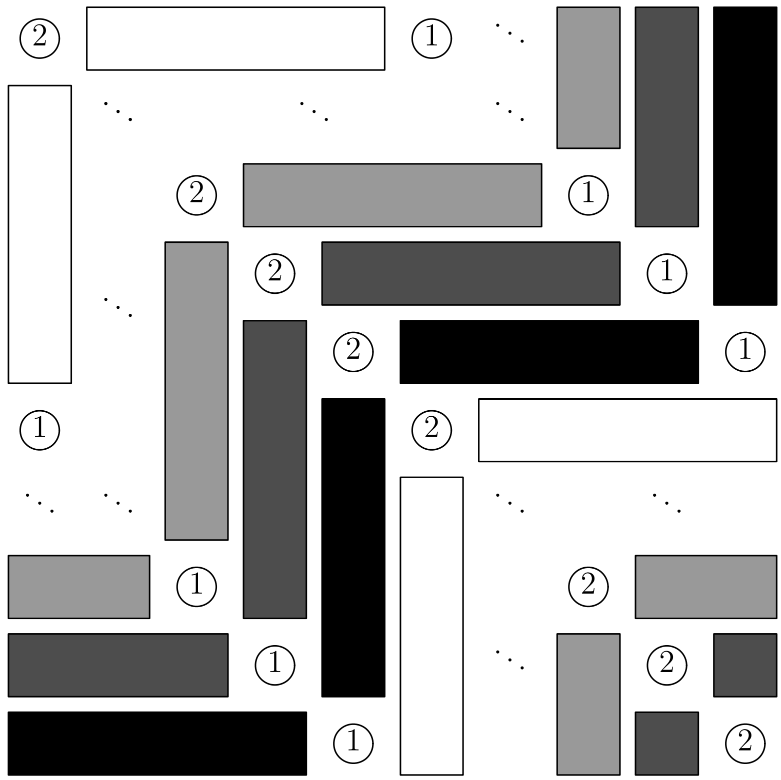

Fig. 10. Jacobsen's construction of a $(\frac{t}{2}+2)$ subcoloring of $K_t \square K_t$ when t is even

PROOF. Assume, without loss of generality, that $m \le n$. As in Jacobson's construction, we use two types of color classes, "stair-step" and "diagonal" (see Figure 11). Each step in a stair-step class consists of a horizonal strip of $nt/(m+n)-1$ elements and a vertical strip of $mt/(m+n)-1$ elements. The element to the right of the horizontal strip is the element below the vertical strip, so locally one diagonal is skipped. In each class there are $m+n$ steps. As in Jacobson's construction, each stair-step class induces a disjoint union of cliques, but we must show that they fit together correctly when shifted up and to the left, as shown in Figure 11. It is perhaps easiest to think of the matrix as occupying the convex hull of $\{(1,1),(1,mt),(nt,1),(nt,mt)\}$ in the integer lattice. Consider the color class B that starts a horizontal step in the lower left corner. The position cyclically to the left of this is $(nt,1)$, the lower right corner. After $\frac{mnt}{m+n}$ shifts, it is at position $(nt-\frac{mnt}{m+n}, \frac{mnt}{m+n}+1)$. This is precisely one step above $(n\frac{nt}{n+m}, n\frac{mt}{n+m})$, which is the end of the nth vertical step in the original color class B, indicated in black. Thus $\frac{mnt}{m+n}$ stair-step classes fit into the matrix without overlaps, but there are still elements of the matrix left uncolored. Examine a column of the partially colored matrix. There are repeated copies of the following formation: a vertical strip of $\frac{mt}{m+n}-1$ entries of the same color, followed by an uncolored entry, followed by $\frac{nt}{m+n}-1$ entries each from a different color horizontal strip, followed by another uncolored entry. Each formation uses $(\frac{mt}{m+n}-1)+1+(\frac{nt}{m+n}-1)+1=t$ entries. There are thus m copies of the formation in each column and therefore $2m$ uncolored

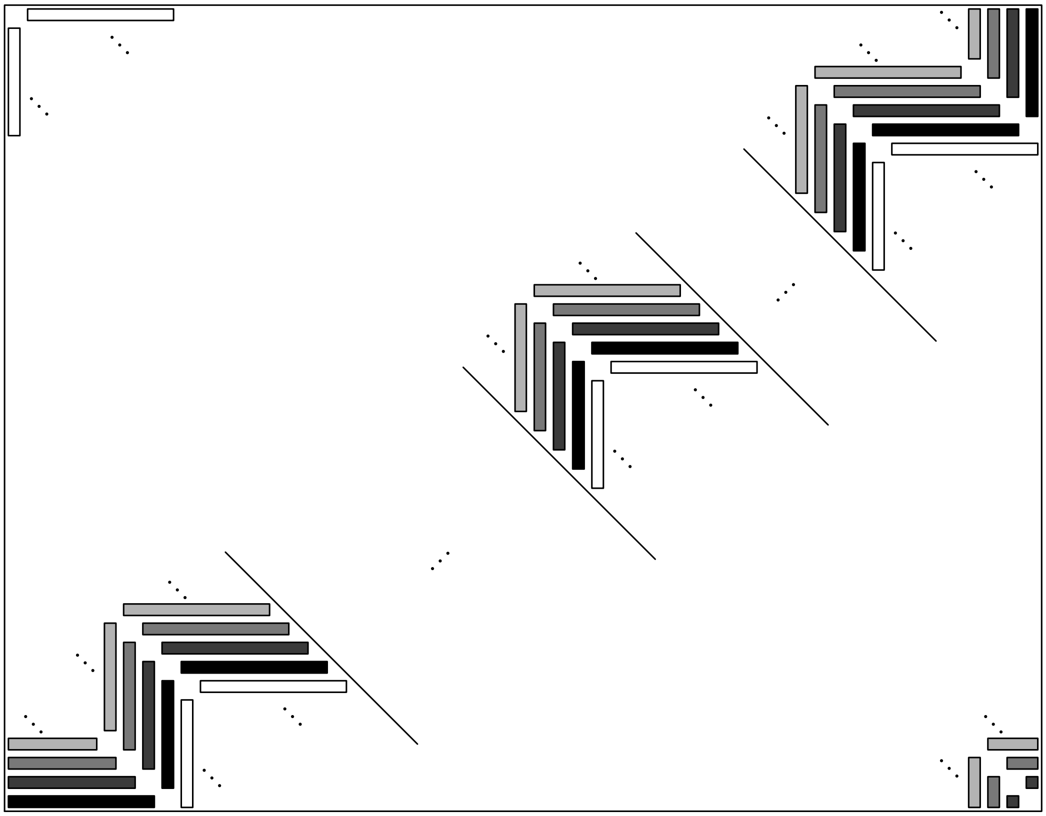

Fig. 11. The general construction of a $\frac{mnt}{m+n} + 2\min\{m,n\}$ coloring of $K_{mt} \,\square\, K_{nt}$

entries in each column. We use an additional $2m$ color classes to color these uncolored entries, using every color class in each column. Since each class is used only once in each column, it is used on a disjoint union of cliques. We have now used $\frac{mnt}{m+n} + 2m = \frac{mnt}{m+n} + 2\min\{m,n\}$ colors to subcolor $K_m \square K_n$. $\square$

In many cases Theorem 15 is much better that the trivial bound $\chi_S(K_m \square K_n) \leq \min\{m,n\}$. For example, with $t = 30$ it shows that $\chi_S(K_{30} \,\square\, K_{60}) \leq 22$. By Proposition 14 at least $\lceil 1800/88 \rceil = 21$ colors are needed.

5 Chromatic and Clique Numbers

We close by considering bounds on subchromatic number in terms of chromatic and clique numbers. Let $\omega(G)$ denote the clique number of G.

As we observed in Remark 1, $\chi_S(G) \leq \chi(G)$. Further, we can form a proper coloring of G from a subcoloring by coloring each class in the subcoloring. Since each class can be properly colored using at most $\omega(G)$ colors we have the following.

Remark 16 *With the preceding notation, $\chi(G) \leq \omega(G)\chi_S(G)$.*

To see that $\chi_S(G) \leq \chi(G)$ is sharp for all clique numbers, we will need the following construction. A G-plex is built from a graph G of order n by taking $n+1$ disjoint copies of G together with n independent vertices that correspond to the vertices of G. We will call these independent vertices the *ground set*. As the ground set is a copy of $V(G)$ we can associate each vertex of the ground set with its $n+1$ counterparts in the copies of G. Make each vertex of the ground set adjacent to each of its $n+1$ counterparts. The resulting graph is known as a *G-plex*.

Theorem 17 *For all integers w and k with $2 \leq w \leq k$ there exists a graph G such that $\omega(G) = w$ and $\chi_S(G) = \chi(G) = k$.*

PROOF. We need only prove the case $w = 2$. For $w > 2$ we may form a graph H with $\omega(G) = 2$ and $\chi_S(H) = \chi(H) = k$, then consider $H \cup K_w$.

To prove the case $w = 2$, the triangle-free graphs, we proceed by induction on k. For the base case, where $k = 2$, consider P_3. For the inductive step we take a triangle-free graph G with $\chi_S(G) = \chi(G) = k$ and construct a triangle-free graph G' with $\chi_S(G') = \chi(G') = k + 1$.

Suppose G has order n. Take a set X of $k(n-1)+1$ independent vertices, and to each subset S of n vertices, attach a G-plex where S is the ground set. Call the resulting graph G'. The upper bound $\chi(G') \leq k+1$ is trivial. We show that every proper subcoloring of G' must use at least $k+1$ colors, thus proving $\chi_S(G') \geq k+1$.

Take any proper subcoloring of G'. If more than k colors are used on the vertices of X, then we are done. If k or fewer colors are used on the vertices of X, then by the pigeonhole principle, some n-subset S of X must be monochromatic. Consider the G-plex attached to S. Again, if more than k colors are used, then we are done. If k or fewer colors are used, then each copy of G must use all k colors, and we can choose at least one vertex from each copy of G that has the same color as S. Since there are $n+1$ copies of G, at least two of these vertices must have the same neighbor in S, and hence the G-plex contains a monochromatic P_3. So the coloring of G' is not a subcoloring and the proof is complete. $\square$

We now consider sharpness in Remark 16. Considering the family of complete graphs, we see that subchromatic number and chromatic number can be arbitrarily far apart. Can the same be said if we restrict the clique number?

Theorem 18 *For all integers w and k with $w \geq 2$ and $k \geq 1$ there exists a graph G such that $\omega(G) = w$, $\chi_S(G) = k$, and $\chi(G) = wk$.*

PROOF. For a fixed w we proceed by induction on k. For the base case with $k = 1$ consider K_w.

Given a graph G satisfying $\omega(G) = w$ and $\chi_S(G) = k$ and $\chi(G) = wk$, we construct a graph G' with $\omega(G') = w$ and $\chi_S(G') = k + 1$ and $\chi(G') = w(k+1)$. Begin with $\alpha = \binom{w(k+1)-1}{wk}(wk-1)+1$ disjoint copies of G, denoted by $G_1, G_2, \ldots, G_\alpha$.

Take a set A consisting of α ordered w-tuples of distinct vertices, one tuple from each copy of G (say $A = \{(v_{1,1}, v_{1,2}, \ldots, v_{1,w}), (v_{2,1}, v_{2,2}, \ldots, v_{2,w}), \ldots, (v_{\alpha,1}, v_{\alpha,2}, \ldots, v_{\alpha,w})\}$ where $v_{i,j} \in V(G_i)$).

Take a complete graph $H_A \cong K_w$ on vertex set $\{1, 2, \ldots, w\}$. Join each vertex $j \in V(H_A)$ to the vertices $\{v_{i,j} | 1 \leq i \leq \alpha\}$ from A. Repeat this procedure for every possible choice of A and denote the resulting graph by G'.

We claim $\omega(G') = w$ and $\chi_S(G) = k + 1$ and $\chi(G) = w(k + 1)$. It is clear that $\omega(G') = w$ since each vertex from each H_A is adjacent to only one vertex from each copy of G, and each pair of vertices from each H_A are adjacent to different vertices in each copy of G. By using k colors on copies of G and one color for all remaining vertices, we see $\chi_S(G') \leq k + 1$. Since $\chi(G') \leq \omega(G')\chi_S(G') \leq w(k + 1)$ we need only show $\chi(G) \geq w(k + 1)$.

To prove $\chi(G) \geq w(k+1)$, consider a proper coloring of G' using the minimum number of colors. If $w(k + 1)$ colors are used on some copy of G, then we are done. Otherwise, label each copy of G with a set of wk colors used on that copy. (Since $\chi(G) = wk$ we are guaranteed there is such a set. If more than wk colors are used on a particular copy of G, then choose any wk-subset of the colors used as the label.) Since there are no more than $\binom{w(k+1)-1}{wk}$ possible labels some label must be used at least wk times on the α copies of G, by the pigeonhole principle. Thus, we can find wk copies of G that use the same set of wk colors (and possibly others) on their vertices. Reorder the copies of G so that these are $G_1, G_2, \ldots, G_{wk}$, and reorder the colors so that the wk colors used are $\{1, 2, \ldots, wk\}$.

For the following, consider the colors as a circular set, so that color $wk + 1$ is actually color 1. Find a set $A = \{(v_{1,1}, v_{1,2}, \ldots, v_{1,w}), (v_{2,1}, v_{2,2}, \ldots, v_{2,w}), \ldots, (v_{\alpha,1}, v_{\alpha,2}, \ldots, v_{\alpha,w})\}$ such that the first w-tuple of vertices (from G_1) receives colors $(1, 2, \ldots, w)$, the second w-tuple of vertices receives colors $(2, 3, \ldots, w+1)$, and so on, up to w-tuple wk (from G_{wk}), which receives colors $(wk, 1, \ldots, wk-1)$. The choices of vertices from other copies of G are not important. Each vertex of the complete graph H_A is adjacent to every color from $\{1, 2, \ldots, wk\}$, so at least w new colors must be used on H_A. Thus, at least $w(k+1)$ colors must be used altogether. $\square$

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