Sperner's Lemma

Def: A *proper labeling* of the vertices of a simplicial subdivision of a d-dimensional simplex assigns the labels $\{0, 1, \ldots, d\}$ to the vertices so that each label is absent from one (d-dimensional) face of the simplex.

Sperner's Lemma: Any properly labeled simplicial subdivision of a d-dimensional simplex contains a completely labeled cell. (That is, a cell whose vertices use every label.)

The Connector Theorem

Def: In a coloring of the vertices of a simplicial complex, a *connector* is a connected, monochromatic subgraph that contains a vertex on each face.

The Connector Theorem:(Hochberg-McDiarmid-Saks) Any d coloring of the vertices of a simplicial subdivision of a d-dimensional simplex contains a connector.

Pouzet's Lemma

Def: In d-space, call the unit vectors along the axes $\{u_1, u_2, \ldots, u_d\}$. A *box* of the integer lattice points is the set of points $\vec{x} = (x_1, x_2, \ldots, x_d)$ satisfying $a_i \leq x_i \leq b_i, 1 \leq i \leq d$. We call two points *neighbors* if their coordinates differ by no more than one in any position.

Pouzet's Lemma: For any mapping f from the lattice points of a d-dimensional box A to the set of unit vectors $\{\pm u_1, \pm u_2, \ldots, \pm u_d\}$ such that $\vec{x} + f(\vec{x})$ is in A for all $\vec{x}$ in A, there exist two neighbors $\vec{x}$ and $\vec{y}$ such that $f(\vec{x}) = -f(\vec{y})$.

The Brouwer Fixed Point Theorem

The Brouwer Fixed Point theorem states that every continuous mapping f from the closed unit box A in d-dimensional space into itself has a point x for which $f(x) = x$.

The Game of Hex

The d-dimensional Game of Hex is played on the lattice points in a d-dimensional box, where two points $\vec{x}$ and $\vec{y}$ are connected if $(\vec{x} - \vec{y})$ or $(\vec{y} - \vec{x})$ is in $\{0, 1\}^d$. (In other words, if they differ by at most one in each coordinate, and every coordinate of $\vec{x}$ is at least as big as the corresponding coordinate of $\vec{y}$ or vice versa.) Each of d players is assigned a distinct color and two opposite faces of the box. Players take turns coloring one point with their own color, the first to create a path of her own color between her faces is the winner. The Theorem of Hex states that there will always be a winner in the Game of Hex.

Proof of Sperner's Lemma

We prove the stronger result that there are an odd number of completely labeled cells. Create a graph G in the dual of the simplicial complex. (That is, this graph has a vertex for each cell of the simplicial complex, and one vertex for the outer region) Connect two vertices in G if the ($(d-1)$-dimensional) face between the cells has precisely the labels $\{0, 1, \ldots, d-1\}$. The vertex corresponding to the outer region of the simplicial complex has odd degree (by induction!). Since each edge in a graph contributes 2 to the total degree, the number of vertices of odd degree in a graph must be even. Thus, there are an odd number of cells whose vertices have odd degree. These cells are completely labeled.

Sperner implies Connector

Assume (for an eventual contradiction) that there is a d coloring of the vertices of a simplicial subdivision of a d-dimensional simplex without a connector. Generate a new coloring in the following manner: Assign the labels $\{0, 1, \ldots, d\}$ to the ($d - 1$ dimensional) faces of the simplex. Label each vertex with the smallest number of an edge it cannot reach on a path of it's own color. This is a proper labeling, so by Sperner's Lemma, there is a completely labeled cell. However, at least two of the vertices of any cell are the same color. All vertices of a cell are adjacent, so we have found two adjacent vertices of the same color that have different smallest edges they can reach, a contradiction.

Connector implies Hex

Given a coloring of a d dimensional box, and the faces each player is trying to connect, add some new vertices. For each player, add one vertex of that players color, and connect it to all vertices on one of that player's faces and to all other new vertices. The result is a d dimensional simplex that has been d colored, so it has a connector. The connector must include vertices from both of the players faces, so that player has won the game of Hex.

Hex implies Pouzet

Given a d-dimensional box A and a function f as in the statement of Pouzet's Lemma, connect the vertices as in the Theorem of Hex. Note that each vertex is connected only to "neighbors" as defined in the statement of Pouzet's Lemma. Assign each dimension a color. Color a vertex $\vec{x}$ of A with the color of the dimension that $f(\vec{x})$ extends in. Assign color i the faces perpendicular to dimension i, and invoke the Theorem of Hex. Since some color "wins" there is a monochromatic path connecting faces perpendicular to some dimension where every vertex $\vec{x}$ on the path has $f(\vec{x})$ in that dimension. Since $f(\vec{x})$ never points outside of A, the first and last vertices on this path have $f(\vec{x})$ pointing in opposite directions. Clearly, there must be two adjacent vertices $\vec{x}$ and $\vec{y}$ on this path where the direction changes. $\vec{x}$ is a neighbor of $\vec{y}$, and $f(\vec{x}) = -f(\vec{y})$.

Pouzet implies Brouwer

Assume (for an eventual contradiction) that there is a continuous function f from the unit box into itself that has no fixed point. Then the function g, equal to the angle of the vector $f(x) - x$, is well defined and continuous. We can thus draw a rectangular grid so that on neighboring (as in Pouzet's Lemma) points of the grid g changes by less that ninety degrees. On such a grid, define a function h that assigns each point the unit vector along the axis closest to $f(x) - x$. Invoking Pouzet's Lemma with $h(x)$, we find two neighboring points x and y where $h(x) = -h(y)$. This is a contradiction, since the definition of g assures us that $f(x) - x$ and $f(y) - y$ differ by less than ninety degrees. Two such vectors cannot be "nearest" to two opposite unit vectors.

Brouwer implies Sperner

Assume (for an eventual contradiction) that there is a proper labeling f of a simplicial subdivision of a simplex that does not have a completely labeled cell. Express every point x in a cell by a convex combination $\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_d v_d$ of the vertices $\{v_0, v_1, \ldots, v_d\}$ of that cell. Create a function g which maps $\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_d v_d$ to $\alpha_0 V_1 + \alpha_1 V_2 + \cdots + \alpha_{d-1} V_d + \alpha_d V_0$ where $\{V_0, V_1, \ldots, V_d\}$ are the (outer) vertices of the simplex. Since there are no completely labeled cells, g is a continuous function that maps every point in the simplex to a point on the faces of the simplex. It is easy to verify that g has no fixed point, which is a contradiction of the Brouwer Fixed Point Theorem.