homework #7 solutions

section 4.1

#7. Show that $\lim_{x\to c} x^3 = c^3$ for all $c \in \mathbb{R}$.

Proof: Let $\epsilon > 0$. Pick $\delta = \min\{1, \frac{\epsilon}{3c^2+3c+1}\}$. Then, for $0 < |x-c| < \delta$, we know

$$|x^3 - c^3| = |x-c||x^2+xc+c^2| < \delta(3c^2+3c+1) \leq \frac{\epsilon}{3c^2+3c+1}(3c^2+3c+1) = \epsilon.$$

#12. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has limit L at 0, and let $a > 0$. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(ax)$ for all $x \in \mathbb{R}$ show that $\lim_{x\to 0} g(x) = L$.

proof: Since $\lim_{x\to 0} f(x) = L$, we know that for an arbitrary $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x| < \delta$ then $|f(x) - L| < \epsilon$. If $0 < |x| < \delta/a$, then $0 < |ax| < \delta$. But, this means $|g(x) - L| = |f(ax) - L| < \epsilon$. Since ϵ is arbitrary, we have shown $\lim_{x\to 0} g(x) = L$.

section 4.2

#9. Let f, g be defined on A to $\mathbb{R}$. Let c be a cluster point of A.
(a) Show that if both $\lim_{x\to c} f$ and $\lim_{x\to c}(f+g)$ exist, then $\lim_{x\to c} g$ exists.

Proof: Assume $\lim_{x\to c} f = N$ and $\lim_{x\to c}(f+g) = M$. We apply Theorem 4.2.4:

$$\lim_{x\to c} g = \lim_{x\to c}(g+f-f) = \lim_{x\to c}(g+f) - \lim_{x\to c} f = M - N.$$

(b) If $\lim f$ and $\lim(fg)$ exists, does it follow that $\lim g$ exists?

Answer: No. Pick $f(x) = x$ and $g(x) = 1/x$ and consider the limits as x approaches 0.

#12. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ and $\lim_{x\to 0} f = L$. Prove $L = 0$. Prove that f has a limit at every point.

Proof: First we show that $L = 0$. Our strategy will be to show that $\lim_{x\to 0} f = L$ implies $\lim_{x\to 0} f = L/2$. Since limits are unique, the only conclusion is $L = 0$.
By assumption, we know that given $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |y| < \delta$, then $|f(y) - L| < \epsilon$. Thus, if $0 < |x| < \delta/2$, then $0 < |2x| < \delta$ and $|f(2x) - L| < \epsilon$. But $|f(2x) - L| = |2f(x) - L| = 2|f(x) - L/2|$. Thus, we now know that if $0 < |x| < \delta/2$, $|f(x) - L/2| < \epsilon/2 < \epsilon$. Since ϵ was arbitrary, this shows that $\lim_{x\to 0} f = L/2$.

Second, we show that f has a limit at every point. From the first part, we know $\lim_{x\to 0} f = 0$. Thus, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |y| < \delta$ then $|f(y)| < \epsilon$. Thus, if $0 < |x-c| < \delta$, then $|f(x-c)| < \epsilon$. But, $|f(x-c)| = |f(x) - f(c)|$. Thus, we know that if $0 < |x-c| < \delta$, $|f(x) - f(c)| < \epsilon$. Since ϵ is arbitrary, we have shown not only that the limit of f at c exists, but we have shown it is equal to $f(c)$.

section 5.1

#3. Assume f is continuous on $[a, b]$ and g is continuous on $[b, c]$ and $f(b) = g(b)$. Define $h(x) = f(x)$ on $[a, b]$ and $h(x) = g(x)$ on $(b, c]$. Show that h is continuous on $[a, c]$.

proof: Given $\epsilon > 0$, we need to show that for every $w \in [a, c]$ there is a γ_w such that if $x \in [a, c]$ and $|x - w| < \gamma_w$, then $|h(x) - h(w)| < \epsilon$.
If $w \in [a, b]$, we know there is $\delta_{w,f}$ such that if $x \in [a, b]$ and $|x - w| < \delta_{w,f}$, then $|f(x) - f(w)| < \epsilon$. Similarly, if $w \in [b, c]$, we know there is $\delta_{w,g}$ such that if $x \in [b, c]$ and $|x - w| < \delta_{w,g}$, then $|g(x) - g(w)| < \epsilon$.
Thus, if $w \in [a, b)$, we pick $\gamma_w = \min\{b - w, \delta_{w,f}\}$ which implies that if $x \in [a, c]$ and $|x - w| < \gamma_w$, then $|h(x) - h(w)| = |f(x) - f(w)| < \epsilon$.
If $w \in (b, c]$, we pick $\gamma_w = \min\{c - w, \delta_{w,g}\}$ which implies that if $x \in [a, c]$ and $|x - w| < \gamma_w$, then $|h(x) - h(w)| = |g(x) - g(w)| < \epsilon$.
If $w = b$ then we pick $\gamma_b = \min\{\delta_{b,f}, \delta_{b,g}\}$ which implies that
(1) if $x \in [a, b]$ and $|x - b| < \gamma_b \leq \delta_{b,f}$, then $|h(x) - h(b)| = |f(x) - f(b)| < \epsilon$.
or
(2) if $x \in [b, c]$ $|x - b| < \gamma_b \leq \delta_{b,g}$, then $|h(x) - h(b)| = |g(x) - g(b)| < \epsilon$.

#6. Let $A \subseteq \mathbb{R}$ and assume $f : A \to \mathbb{R}$ is continuous at $c \in A$. Show that for any $\epsilon > 0$ there exists a neighborhood $V_\delta(c)$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in (A \cap V_\delta(c)$.

proof: Given $\epsilon > 0$. Since f is continuous at c, we know there exists $\delta > 0$ such that if $xinA$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon/2$. Thus, for all $x, y \in A \cap V_\delta(c)$,

$$|f(x) - f(y)| = |f(x) - f(c) + f(c) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$$

which is what we wanted to show.

section 5.2

#10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $P = \{x \in \mathbb{R} | f(x) > 0\}$. If $c \in P$ show that there exists a neighborhood $V_\delta(c) \subseteq P$.

proof: By assumption, we know $f(c) > 0$. Thus, pick $\epsilon = f(c)/2$. Since f is continuous at c, we know there exists $\delta > 0$ such that if $x \in V_\delta(c)$, then $f(x) \in V_\epsilon(f(c))$. Thus, for all $x \in V_{delta}(c)$, $0 < f(c)/2 < f(x) < 3f(c)/2$. Thus, $V_{delta}(c) \subseteq P$.