homework #3 solutions

Section 2.4

#2. If \( S = \{1/n - 1/m : n, m \in \mathbb{N}\} \), find \( \inf S \) and \( \sup S \).

Answer. We claim \( \inf S = -1 \). Let \( 1/n - 1/m \) be an arbitrary element in \( S \). Then, \( 1/n - 1/m \geq 1/n - 1 > -1 \). So \(-1\) is a lower bound for \( S \).

Let \( \epsilon > 0 \). By Corollary 2.4.5, there exists \( n_0 \in \mathbb{N} \) such that \( 1/n_0 < \epsilon \). Now, \( 1/n_0 - 1 < \epsilon - 1 = -1 + \epsilon \) and \( 1/n_0 - 1 \in S \). Thus, \(-1 = \inf S \).

We claim \( \sup S = 1 \). Note that \( S = -S \) and apply homework problem #5 from section 2.3 in which you proved \( \inf S = -\sup\{-s : s \in S\} \). We get \(-1 = \inf S = -\sup\{-s : s \in S\} = -\sup S \) which implies \( \sup S = 1 \).

#4.b. Let \( S \) be a nonempty bounded set in \( \mathbb{R} \). Let \( b < 0 \) and let \( bS = \{bs : s \in S\} \).

Prove that \( \inf bS = b\sup S \) and \( \sup bS = b\inf S \).

Proof: Let \( S \) be a nonempty bounded set in \( \mathbb{R} \). Thus \( S \) has an infimum and a supremum. Let \( v = \sup S \). We need to show that \( bv = \inf S \). Let \( bs \) be an arbitrary element of \( bS \). Then, \( s \in S \) and so \( s \leq v \). But this implies that \( bs \geq bv \). Thus, \( bv \) is a lower bound for \( bS \). Let \( r \) be an arbitrary element of \( \mathbb{R} \) such that \( bv < r \). Then \( v > r/b \). Since \( v = \sup S \), there exists an element \( s_0 \in S \) such that \( s_0 > r/b \). But this implies that for the element \( bs_0 \in bS \), \( bs_0 < r \). Thus, we have shown that \( bv = \inf bS = b\sup S \).

Let \( w = \inf S \). We need to show that \( bw = \sup S \). Let \( bs \) be an arbitrary element of \( bS \). Then, \( s \in S \) and so \( s \geq v \). But this implies that \( bs \leq bw \). Thus, \( bw \) is a lower bound for \( bS \). Let \( r \) be an arbitrary element of \( \mathbb{R} \) such that \( bw > r \). Then \( w < r/b \). Since \( w = \inf S \), there exists an element \( s_0 \in S \) such that \( s_0 < r/b \). But this implies that, for the element \( bs_0 \in bS \), \( bs_0 > r \). Thus, we have shown that \( bw = \sup bS = b\inf S \).

# 6. Let \( A \) and \( B \) be bounded nonempty subsets of \( \mathbb{R} \), and let \( A + B = \{a + b : a \in A \text{ and } b \in B\} \). Prove that \( \sup(A+B) = \sup A + \sup B \) and \( \inf(A+B) = \inf A + \inf B \).

Proof: Let \( A \) and \( B \) be bounded nonempty subsets of \( \mathbb{R} \). Then we know they both
have an infimum and a supremum. Let \( u_A \) and \( u_B \) be the supremum of \( A \) and \( B \) respectively. We want to show that \( u_A + u_B = \sup(A + B) \). Let \( c \) be an arbitrary element of \( A + B \). Then there exist \( a \in A \) and \( b \in B \) such that \( c = a + b \). Now, \( c = a + b \leq u_A + b \leq u_A + u_B \) where the first inequality follows from \( u_A = \sup A \) and the second follows from \( u_B = \inf B \). This shows that \( u_A + u_B \) is an upper bound for \( A + B \). Let \( \epsilon > 0 \). We need to find an element \( c_0 \in A + B \) so that \( c_0 > u_A + u_B - \epsilon \). We know there exist elements \( a_0 \in A \) and \( b_0 \in B \) such that \( a_0 > u_A - \epsilon/2 \) and \( b_0 > u_B - \epsilon/2 \). Thus we pick \( c_0 = a_0 + b_0 > u_A - \epsilon/2 + u_B - \epsilon/2 = u_A + u_B - \epsilon \). Thus, we have shown \( \sup(A + B) = \sup A + \sup B \).

Let \( u_A \) and \( u_B \) be the infimum of \( A \) and \( B \) respectively. We want to show that \( w_A + w_B = \inf(A + B) \). Let \( c \) be an arbitrary element of \( A + B \). Then there exist \( a \in A \) and \( b \in B \) such that \( c = a + b \). Now, \( c = a + b \geq w_A + b \geq w_A + w_B \) where the first inequality follows from \( u_A = \inf A \) and the second follows from \( u_B = \inf B \). This shows that \( u_A + u_B \) is a lower bound for \( A + B \). Let \( \epsilon > 0 \). We need to find an element \( c_0 \in A + B \) so that \( c_0 < w_A + w_B + \epsilon \). We know there exist elements \( a_0 \in A \) and \( b_0 \in B \) such that \( a_0 < w_A + \epsilon/2 \) and \( b_0 < w_B + \epsilon/2 \). Thus we pick \( c_0 = a_0 + b_0 < w_A + \epsilon/2 + w_B + \epsilon/2 = w_A + w_B + \epsilon \). Thus, we have shown \( \inf(A + B) = \inf A + \inf B \).

# 10. Let \( X \) and \( Y \) be nonempty sets and let \( h : X \times Y \to R \) have bounded range in \( R \). Let \( f : X \to R \) and \( g : Y \to R \) be defined by

\[
f(x) = \sup \{ h(x, y) : y \in Y \}, \quad g(y) = \inf \{ h(x, y) : x \in X \}.
\]

Prove \( \sup \{ g(y) : y \in Y \} \leq \inf \{ f(x) : x \in X \} \).

Proof. Since \( h \) is bounded, we know \( f \) and \( g \) are well-defined and are themselves bounded. Let \( u = \sup g \). In order to show that \( u \leq \inf f \) it is enough to show that \( u \leq f(x) \) for all \( x \in X \). Let \( \epsilon > 0 \). We know there exists a \( y_0 \in Y \) such that \( u - \epsilon < g(y_0) \). But, by the definition of \( g \), \( g(y_0) \leq h(x, y_0) \) for all \( x \in X \). But, by the definition of \( f \), \( f(x) \geq h(x, y_0) \) for all \( x \in X \). Thus, for every \( \epsilon > 0 \) and every \( x \in X \), \( u - \epsilon < g(y_0) \leq h(x, y_0) \leq f(x) \). Thus, \( u \leq f(x) \) for every \( x \in X \) which is what we needed to show.

Section 2.5

# 6. If \( I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots \) is a nested sequence of intervals and if \( I_n = [a_n, b_n] \), show \( a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \) and \( b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots \).
Proof. We will show that $a_k \leq a_{k+1}$ for all $k \in \mathbb{N}$. We know $a_{k+1} \in I_{k+1} \subseteq I_k$ by assumption. So, $a_{k+1} \in I_k$. But, by definition of $I_k$, $a_k \leq a_{k+1} \leq b_k$. So, $a_k \leq a_{k+1}$.

Next we will show that $b_k \geq b_{k+1}$ for all $k \in \mathbb{N}$. We know $b_{k+1} \in I_{k+1} \subseteq I_k$ by assumption. So, $b_{k+1} \in I_k$. But, by definition of $I_k$, $a_k \leq b_{k+1} \leq b_k$. So, $b_k \geq b_{k+1}$.

# 7. Let $I_n = [0, 1/n]$ for $n \in \mathbb{N}$. Show $\cap_{n=1}^{\infty} I_n = \{0\}$.

Proof. By definition, $0 \in [0, 1/n] = I_n$ for all $n \in \mathbb{N}$. So $\{0\} \subseteq \cap_{n=1}^{\infty} I_n$. Let $x \in \cap_{n=1}^{\infty} I_n = \{0\}$. We need to show that $x = 0$. By our choice of $x$, we know $0 \leq x \leq 1/n$ for all $n \in \mathbb{N}$. But we know that for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $1/n_\epsilon < \epsilon$. Thus, for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $0 \leq x \leq 1/n_\epsilon < \epsilon$ which implies $x = 0$.

# 9. Let $K_n = (n, \infty)$ for $n \in \mathbb{N}$. Show $\cap_{n=1}^{\infty} K_n = \emptyset$.

Proof. We will proceed by contradiction. Assume there exists a real number $x \in \cap_{n=1}^{\infty} K_n$. By the Archimedean Property, we know there exists a natural number $n_0$ such that $x < n_0$. But this implies that $x \not\in K_{n_0}$. This contradicts our choice of $x$. Thus, there is no real number $x \in \cap_{n=1}^{\infty} K_n$ and so $\cap_{n=1}^{\infty} K_n = \emptyset$. 