homework #3 solutions

Section 2.4

#2. If $S = \{1/n - 1/m : n, m \in N\}$, find $\inf S$ and $\sup S$.

Answer. We claim $\inf S = -1$. Let $1/n - 1/m$ be an arbitrary element in S. Then, $1/n - 1/m \geq 1/n - 1 > -1$. So -1 is a lower bound for S.
Let $\epsilon > 0$. By Corollary 2.4.5, there exists $n_0 \in N$ such that $1/n_0 < \epsilon$. Now, $1/n_0 - 1 < \epsilon - 1 = -1 + \epsilon$ and $1/n_0 - 1 \in S$. Thus, $-1 = \inf S$.

We claim $\sup S = 1$. Note that $S = -S$ and apply homework problem #5 from section 2.3 in which you proved $\inf S = -\sup\{-s : s \in S\}$. We get $-1 = \inf S = -\sup\{-s : s \in S\} = -\sup S$ which implies $\sup S = 1$.

#4.b. Let S be a nonempty bounded set in R. Let $b < 0$ and let $bS = \{bs : s \in S\}$. Prove that $\inf bS = b \sup S$ and $\sup bS = b \inf S$.

Proof: Let S be a nonempty bounded set in R. Thus S has an infimum and a supremum. Let $v = \sup S$. We need to show that $bv = \inf S$. Let bs be an arbitrary element of bS. Then, $s \in S$ and so $s \leq v$. But this implies that $bs \geq bv$. Thus, bv is a lower bound for bS. Let r be an arbitrary element of R such that $bv < r$. Then $v > r/b$. Since $v = \sup S$, there exists an element $s_0 \in S$ such that $s_0 > r/b$. But this implies that for the element $bs_0 \in bS$, $bs_0 < r$. Thus, we have shown that $bv = \inf bS = b \sup S$.

Let $w = \inf S$. We need to show that $bw = \sup S$. Let bs be an arbitrary element of bS. Then, $s \in S$ and so $s \geq v$. But this implies that $bs \leq bw$. Thus, bw is a lower bound for bS. Let r be an arbitrary element of R such that $bw > r$. Then $w < r/b$. Since $w = \inf S$, there exists an element $s_0 \in S$ such that $s_0 < r/b$. But this implies that, for the element $bs_0 \in bS$, $bs_0 > r$. Thus, we have shown that $bw = \sup bS = b \inf S$.

6. Let A and B be bounded nonempty subsets of R, and let $A + B = \{a + b : a \in A and b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ and $\inf(A + B) = \inf A + \inf B$.

Proof: Let A and B be bounded nonempty subsets of R. Then we know they both have an infimum and a supremum. Let u_A and u_B be the supremum of A and B respectively. We want to show that $u_A + u_B = \sup(A+B)$. Let c be an arbitrary element of $A+B$. Then there exist $a \in A$ and $b \in B$ such that $c = a+b$. Now, $c = a+b \leq u_A+b \leq u_A+u_B$ where the first inequality follows from $u_A = \sup A$ and the second follows from $u_B = \sup B$. This shows that u_A+u_B is an upper bound for $A+B$. Let $\epsilon > 0$. We need to find an element $c_0 \in A+B$ so that $c_0 > u_A+u_b-\epsilon$. We know there exist elements $a_0 \in A$ and $b_0 \in B$ such that $a_0 > u_A - \epsilon/2$ and $b_0 > u_B - \epsilon/2$. Thus we pick $c_0 = a_0 + b_0 > u_A - \epsilon/2 + u_B - \epsilon/2 = u_A + u_B - \epsilon$. Thus, we have shown $\sup(A+B) = \sup A + \sup B$.

Let u_A and u_B be the infimum of A and B respectively. We want to show that $w_A + w_B = \inf(A+B)$. Let c be an arbitrary element of $A+B$. Then there exist $a \in A$ and $b \in B$ such that $c = a+b$. Now, $c = a+b \geq w_A+b \geq w_A+w_B$ where the first inequality follows from $u_A = \inf A$ and the second follows from $u_B = \inf B$. This shows that u_A+u_B is a lower bound for $A+B$. Let $\epsilon > 0$. We need to find an element $c_0 \in A+B$ so that $c_0 < w_A+w_b+\epsilon$. We know there exist elements $a_0 \in A$ and $b_0 \in B$ such that $a_0 < w_A + \epsilon/2$ and $b_0 < w_B + \epsilon/2$. Thus we pick $c_0 = a_0 + b_0 < w_A + \epsilon/2 + w_B + \epsilon/2 = u_A + u_B + \epsilon$. Thus, we have shown $\inf(A+B) = \inf A + \inf B$.

10. Let X and Y be nonempty sets and let $h : X \times Y \to R$ have bounded range in R. Let $f : X \to R$ and $g : Y \to R$ be defined by

$$f(x) = \sup\{h(x,y) : y \in Y\},\ g(y) = \inf\{h(x,y) : x \in X\}.$$

Prove $\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}$.

Proof. Since h is bounded, we know f and g are well-defined and are themselves bounded. Let $u = \sup g$. In order to show that $u \leq \inf f$ it is enough to show that $u \leq f(x)$ for all $x \in X$. Let $\epsilon > 0$. We know there exists a $y_0 \in Y$ such that $u - \epsilon < g(y_0)$. But, by the definition of g, $g(y_0) \leq h(x, y_0)$ for all $x \in X$. But, by the definition of f, $f(x) \geq h(x, y_0)$ for all $x \in X$. Thus, for *every* $\epsilon > 0$ and every $x \in X$, $u - \epsilon < g(y_0) \leq h(x, y_0) \leq f(x)$. Thus, $u \leq f(x)$ for every $x \in X$ which is what we needed to show.

Section 2.5

6. If $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ is a nested sequence of intervals and if $I_n = [a_n, b_n]$, show $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq \cdots$.

Proof. We will show that $a_k \leq a_{k+1}$ for all $k \in N$. We know $a_{k+1} \in I_{k+1} \subseteq I_k$ by assumption. So, $a_{k+1} \in I_k$. But, by definition of I_k, $a_k \leq a_{k+1} \leq b_k$. So, $a_k \leq a_{k+1}$. Next we will show that $b_k \geq b_{k+1}$ for all $k \in N$. We know $b_{k+1} \in I_{k+1} \subseteq I_k$ by assumption. So, $b_{k+1} \in I_k$. But, by definition of I_k, $a_k \leq b_{k+1} \leq b_k$. So, $b_k \geq b_{k+1}$.

7. Let $I_n = [0, 1/n]$ for $n \in N$. Show $\cap_{n=1}^{\infty} I_n = \{0\}$.

Proof. By definition, $0 \in [0, 1/n] = I_n$ for all $n \in N$. So $\{0\} \subseteq \cap_{n=1}^{\infty} I_n$. Let $x \in \cap_{n=1}^{\infty} I_n = \{0\}$. We need to show that $x = 0$. By our choice of x, we know $0 \leq x \leq 1/n$ for all $n \in N$. But we know that for every $\epsilon > 0$, there exists $n_\epsilon \in N$ such that $1/n_\epsilon < \epsilon$. Thus, for every $\epsilon > 0$ there exists $n_\epsilon \in N$ such that $0 \leq x \leq 1/n_\epsilon < \epsilon$ which implies $x = 0$.

9. Let $K_n = (n, \infty)$ for $n \in N$. Show $\cap_{n=1}^{\infty} K_n = \emptyset$.

Proof. We will proceed by contradiction. Assume there exists a real number $x \in \cap_{n=1}^{\infty} K_n$. By the Archimedean Property, we know there exists a natural number n_0 such that $x < n_0$. But this implies that $x \notin K_{n_0}$. This contradicts our choice of x. Thus, there is no real number $x \in \cap_{n=1}^{\infty} K_n$ and so $\cap_{n=1}^{\infty} K_n = \emptyset$.