Spanning Trees

CS 311 Data Structures and Algorithms
Lecture Slides
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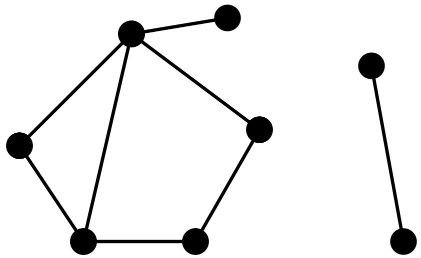
The Rest of the Course
Overview

Final Topics

✓ ▪ External Data
  - Previously, we dealt only with data stored in memory.
  - Suppose, instead, that we wish to deal with data stored on an external device, accessed via a relatively slow connection and available in chunks (data on a disk, for example).
  - How does this affect the design of algorithms and data structures?

(part) ▪ Graph Algorithms
  - A **graph** models relationships between pairs of objects.
  - This is a very general notion. Algorithms for graphs often have very general applicability.



Drawing of
a Graph

This usage of "graph" has nothing to do with the graph of a function. It is a different definition of the word.

Module Overview
Graph Algorithms

Topics

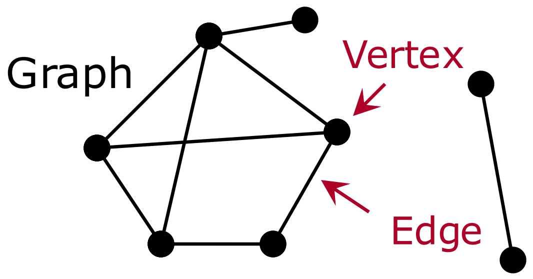
- ✓ Introduction to Graphs
- ✓ Graph Traversals
- Spanning Trees
- Other Graph Topics

Review

Introduction to Graphs [1/3]

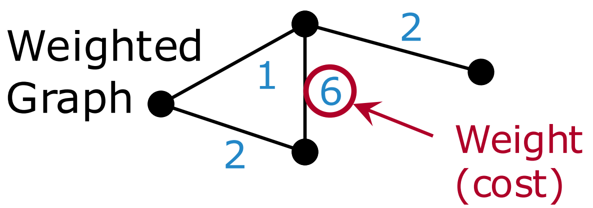
A **graph** consists of **vertices** and **edges**.

- An edge joins two vertices: its **endpoints**.
- 1 **vertex**, 2 vertices (Latin plural).
- Two vertices joined by an edge are **adjacent**; each is a **neighbor** of the other.



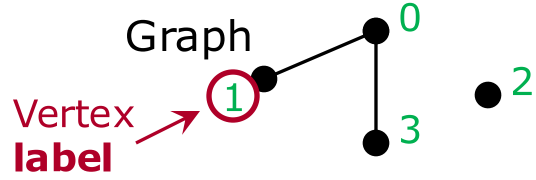
In a **weighted graph**, each edge has a **weight** (or **cost**).

- The weight is the resource expenditure required to use that edge.
- We typically choose edges to minimize the total weight of some kind of collection.



Introduction to Graphs [2/3]

Two common ways to represent graphs.



Adjacency matrix. 2-D array of 0/1 values.

- "Are vertices *i*, *j* adjacent?" in Θ(1) time.
- Finding all neighbors of a vertex is slow for large, sparse graphs.

Adjacency Matrix

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency lists. List of lists (arrays?). List *i* holds neighbors of vertex *i*.

- "Are vertices *i*, *j* adjacent?" in Θ(log *N*) time if lists are sorted arrays; Θ(*N*) if not.
- Finding all neighbors can be faster.

N: the number of vertices.

Adjacency Lists

0: 1, 3
1: 0
2:
3: 0

Review
Introduction to Graphs [3/3]

When we analyze the efficiency of graph algorithms, we consider *both* the number of vertices and the number of edges.

- N = number of vertices
- M = number of edges

When analyzing efficiency, we consider adjacency matrices & adjacency lists separately.

The *total* size of the input is:

- For an adjacency matrix: N^2. So $\Theta(N^2)$.
- For adjacency lists: $N + 2M$. So $\Theta(N + M)$.

Some particular algorithm might have order (say) $\Theta(N + M \log N)$.

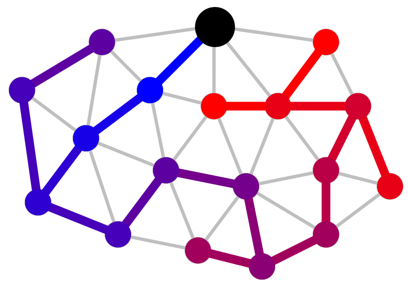
Graph Traversals — Introduction

To **traverse** a graph means to visit each vertex once.

Two important graph traversals:

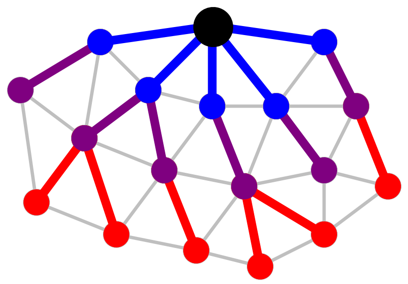
Depth-first search (**DFS**)

Walk along edges, visiting vertices as we go. When there is nowhere new to go, backtrack.



Breadth-first search (**BFS**)

Proceed along all edges from a vertex, visiting each of its neighbors. Then visit its neighbors’ neighbors, etc.

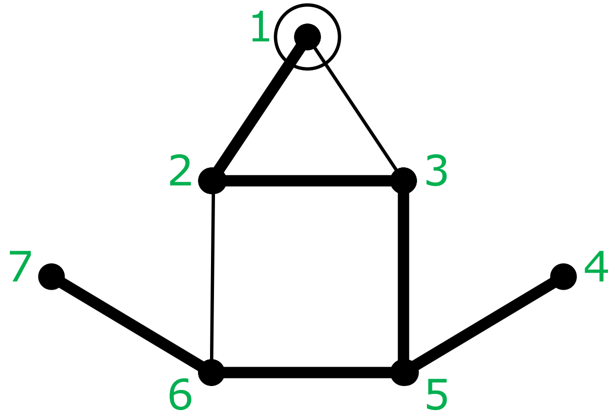


In both cases, we prefer to visit lower-numbered vertices first.

Review
Graph Traversals — DFS [1/2]

DFS has a natural recursive formulation:

- Given a *start* vertex, visit it, and mark it as visited.
- For each of the start vertex's neighbors:
  - If this neighbor is unvisited, then do a DFS with this neighbor as the start vertex.



DFS: 1, 2, 3, 5, 4, 6, 7

Recursion can be eliminated with a Stack—of course. But we can be more intelligent than the brute-force method.

Review

Graph Traversals — DFS [2/2]

Algorithm DFS [non-recursive]

- Mark all vertices as unvisited.
- For each vertex:
  - Do algorithm DFS' with this vertex as *start*.

Algorithm DFS' [non-recursive]

- Set Stack to empty.
- Push *start* vertex on Stack.
- Repeat while Stack is non-empty:
  - Pop top of Stack.
  - If this vertex is not visited, then:
    - Visit it.
    - Push its not-visited neighbors on the Stack.

This part is all we need, if the graph is **connected** (all one piece). The above makes it work for all graphs, including **disconnected** graphs.

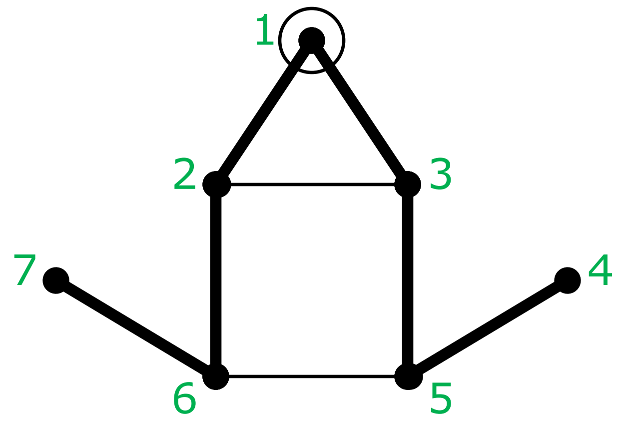
DONE

- Based on the above, write a non-recursive function to do a DFS on a graph, given adjacency lists.

See `graph_traverse.cpp`.

Graph Traversals — BFS

We can easily do a BFS manually by keeping track of those vertices for which we have looked at all neighbors.



BFS: 1, 2, 3, 6, 5, 7, 4

DONE

- Modify our DFS function to do BFS.
  - BFS reverses the priority of neighbors of vertex visited most recently vs. neighbors of vertices visited earliest.
  - Thus, replace the Stack with a Queue.

See `graph_traverse.cpp`.

Review
Graph Traversals — Efficiency

We can analyze the DFS and BFS algorithms by looking, first, at how much processing is done for each vertex.

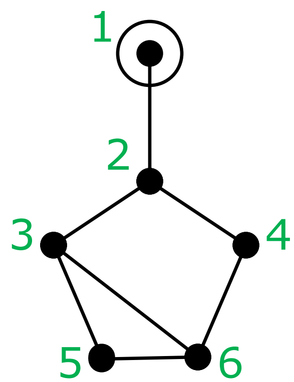
Second, we look at how much is done for each edge when the graph is given via adjacency lists, or each matrix row when the graph is given via an adjacency matrix.

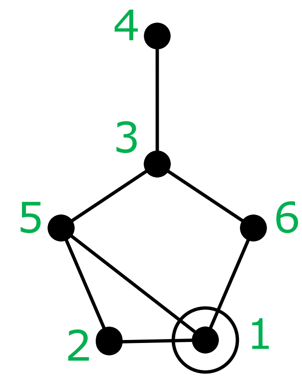
When given adjacency lists, each algorithm is $\Theta(N + M)$. And when given an adjacency matrix, each algorithm is $\Theta(N^2)$.

In all cases, the running time is of the same order as the total size of the input. So there cannot be DFS/BFS algorithms that are much faster than those we covered.

Review
Graph Traversals — Try It! [1/2]

For each graph below, write the order in which the vertices will be visited in a DFS and in a BFS.

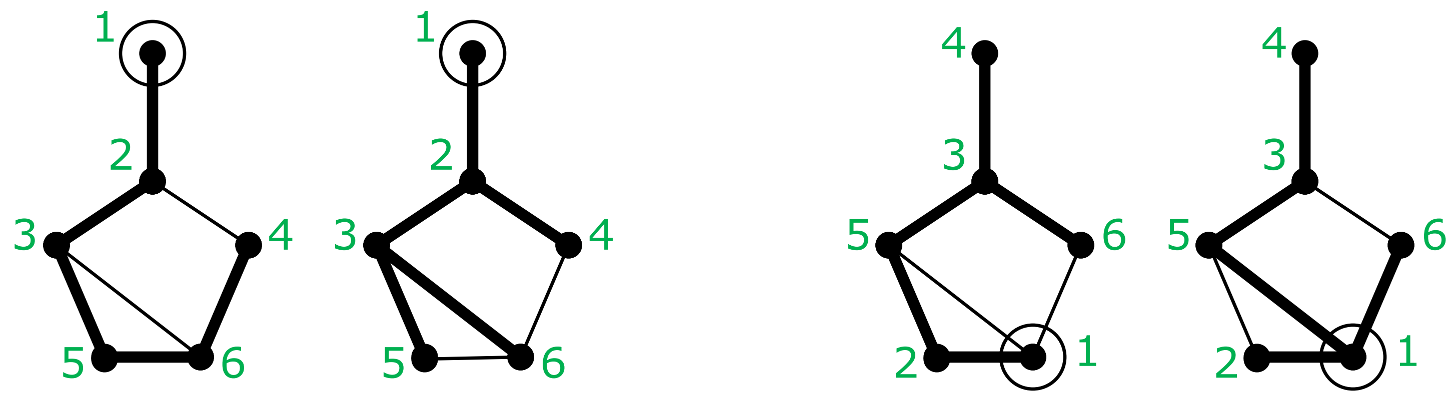




Answers on next slide.

Review
Graph Traversals — Try It! [2/2]

For each graph below, write the order in which the vertices will be visited in a DFS and in a BFS.



Answers

DFS: 1, 2, 3, 5, 6, 4

BFS: 1, 2, 3, 4, 5, 6

DFS: 1, 2, 5, 3, 4, 6

BFS: 1, 2, 5, 6, 3, 4

Spanning Trees

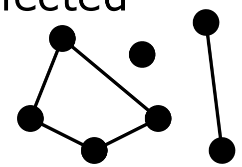
Spanning Trees
Introduction [1/3]

A **tree** is a graph that:

- Is **connected** (all one piece).
- Has no **cycles**.

Here, "tree" does *not* mean "rooted tree".

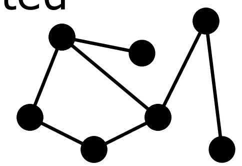
Disconnected Graph



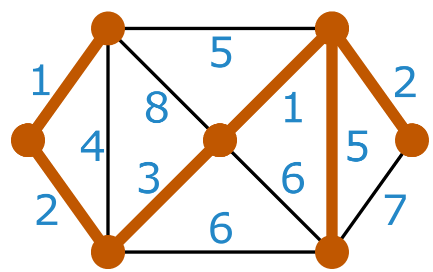
A **spanning tree** in a graph *G* is a tree that:

- Includes only vertices and edges of *G*.
- Includes *all* vertices of *G*.

Connected Graph



Fact. Every connected graph has a spanning tree.

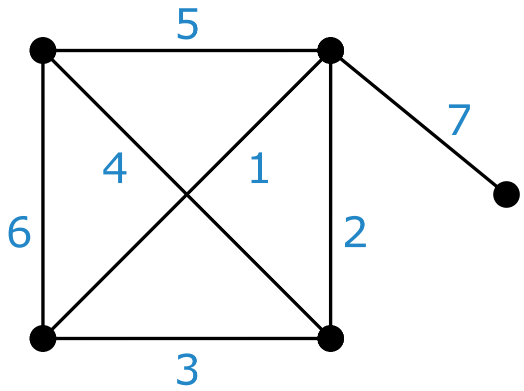


An important problem: given a weighted graph, find a **minimum spanning tree**—a spanning tree of minimum total weight.

There are several nice algorithms that solve this problem.

Try to find a minimum spanning tree in the following weighted graph. Draw what you find, and determine its total weight.

Blue numbers are edge weights.

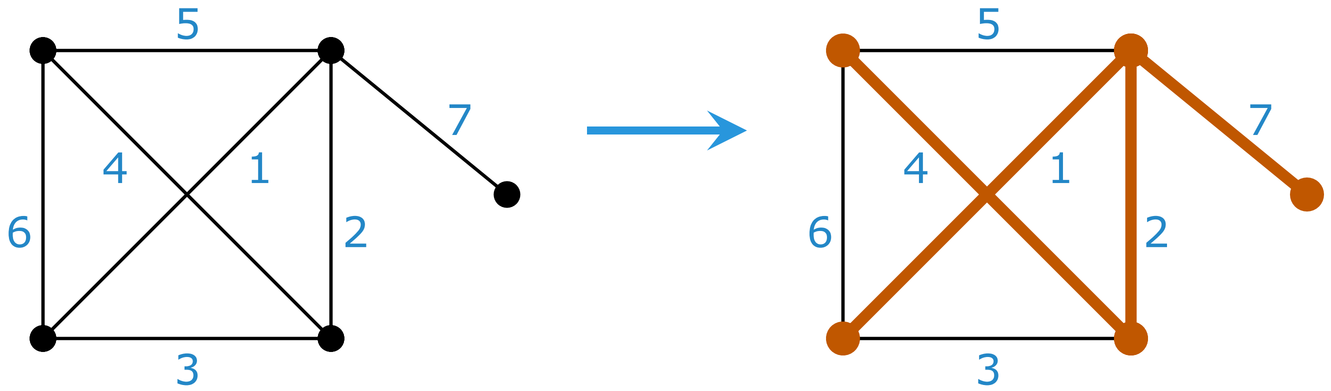


Answer on next slide.

Spanning Trees
Introduction [3/3] (Try It!)

Try to find a minimum spanning tree in the following weighted graph. Draw what you find, and determine its total weight.

Blue numbers are edge weights.



Total weight = 1 + 2 + 4 + 7 = 14

Spanning Trees
Greedy Algorithms

We can find a minimum spanning tree using a *greedy* algorithm.

A **greedy** algorithm is "shortsighted". It proceeds in a series of choices, each based on *what is known at the time*. Choices are:

- **Feasible.** Each makes sense.
- **Locally optimal.** Best possible based on current information.
- **Irrevocable.** Once a choice is made, it is permanent. A greedy algorithm never backtracks.

Being greedy is usually *not* a good way to get correct answers.

However, in the cases when being greedy gives correct results, it tends to be *very fast*.

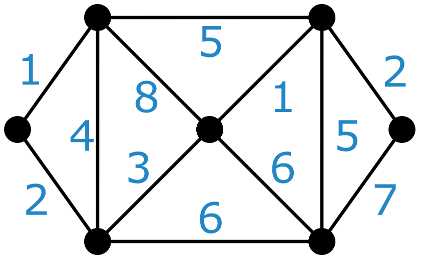
This idea is not just for minimum spanning trees; there are many useful greedy algorithms. *See CS 411.*

Spanning Trees
Prim's Algorithm — Idea

Here is an idea for a greedy algorithm to find a minimum spanning tree in a connected weighted graph.

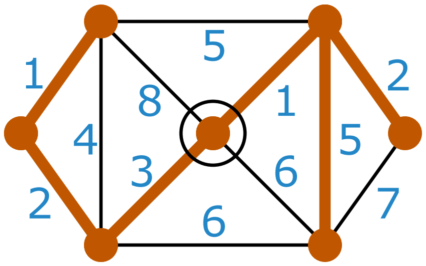
- One vertex is specified as *start*.
- As the algorithm proceeds, we add edges to a tree. Using these edges, we are able to reach more and more vertices from *start*.
- Procedure. Repeatedly add the lowest-weight edge from a reachable vertex to a non-reachable vertex, until all vertices are reachable.

This idea leads to a—correct!—greedy algorithm to find a minimum spanning tree: **Prim's Algorithm**, also called the **Prim-Jarník Algorithm**. [V. Jarník 1930, R. C. Prim 1957, E. Dijkstra 1959]



Prim’s Algorithm

- Given: Connected graph, weights on the edges; one vertex is *start*.
- Returns: Edge set of a minimum spanning tree.
- Procedure:
  - Mark all vertices as not-reachable.
  - Set edge set of tree to empty.
  - Mark *start* vertex as reachable.
  - Repeat while there exist not-reachable vertices:
    - Find lowest weight edge joining a reachable vertex to a not-reachable vertex.
    - Add this edge to the tree.
    - Mark the not-reachable endpoint of this edge as reachable.
  - Return edge set of tree.



It is not obvious that Prim’s Algorithm correctly finds a minimum spanning tree. *But it does!*

Spanning Trees
Prim's Algorithm — Issues [1/2]

How can we efficiently find the lowest-weight edge between reachable and a not-reachable vertices?

- Use a Priority Queue holding edges, ordered by weight and implemented as a Minheap. So we do getFront & delete on the edge of *least* weight.
- Insert edges that join reachable & not-reachable vertices.
- When to insert? When marking a vertex as reachable, insert into the PQ all edges from this vertex to not-reachable neighbors.
- This means that, for each edge in the PQ, *at some point*, it joined reachable & non-reachable vertices.
- When getting an edge from the PQ, check to be sure it *still* joins reachable & not-reachable vertices. If not, skip it.
- When the PQ is empty, quit.

It was mentioned a few weeks ago that we would eventually cover an application of a Priority Queue. This is it!

Spanning Trees
Prim's Algorithm — Issues [2/2]

How can we represent a weighted graph?

- Use something like an adjacency matrix, but instead of storing 0/1, store weights.
- Also allow each entry in the matrix to have a special value meaning *no edge*.
- We may wish to have adjacency lists as well, for efficiency.

When the spanning tree is finished, the PQ may not be empty yet.

- Easy optimization: track the number of non-reachable vertices. Stop when this is zero.
- Equivalently, stop when the tree has $N-1$ edges, where N is the number of vertices in the graph.

Spanning Trees
Prim's Algorithm — CODE

TO DO

- Implement Prim's Algorithm.
  - Use `std::priority_queue` to find the lowest-weight edge.

Done. See `prim.cpp`*.*

Prim's Algorithm — Efficiency [1/3]

What is the order of our implementation of Prim's Algorithm?

- It does something with each vertex.
- It does something with each edge.
- The latter may involve insertion & deletion in a Priority Queue—implemented using a Binary Heap.
- We do a lot of these, so we can ignore the "amortized" in the time required for the Priority Queue insertion.

Result: $\Theta(N + M \log M)$.

For a connected graph, we have $M \geq N - 1$.

So: $\Theta(M \log M)$.

Spanning Trees
Prim's Algorithm — Efficiency [2/3]

Our Prim's Algorithm implementation is $\Theta(M \log M)$.

But there are implementations that are a bit more efficient.

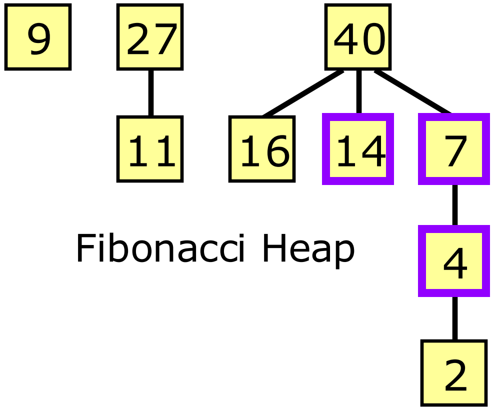
Idea #1. Have the Priority Queue hold vertices, ordered by cost, not edges.

- The cost of a non-reachable vertex is the cost of the least weight edge from it to a reachable vertex—or ∞, if there is no such edge.
- When we mark a vertex as reachable, we may need to update the cost of some of the vertices in the Priority Queue.
- So we need a structure with more operations than a Priority Queue.
- We can use a Binary Heap (Minheap) in which each vertex in the graph keeps track of where it is in the Heap. When we reduce the cost of a vertex, we do a sift-up on that vertex.
- This operation is called **decrease-key** (or **increase-key** for a Maxheap). It is logarithmic time for a Binary Heap.

Spanning Trees
Prim’s Algorithm — Efficiency [3/3]

Idea #2 (used with Idea #1). Replace the Binary Heap with a *Fibonacci Heap*. [M. L. Fredman & R. E. Tarjan 1987]

- A **Fibonacci Heap** is a data structure similar to a Binary Heap, but:
  - Insert is Θ(1).
  - Increase-key (decrease-key for a Minheap) is amortized Θ(1).
  - Delete is amortized Θ(log n).
  - No array representation is used.



Fibonacci Heap

- Prim’s Algorithm goes through every vertex and every edge, so we can ignore the “amortized” when finding its running time.
- For each edge, we do a fixed number of operations that are all (possibly amortized) constant-time.
- For each vertex, we do a fixed number of operations that are all (possibly amortized) constant-time or logarithmic-time.

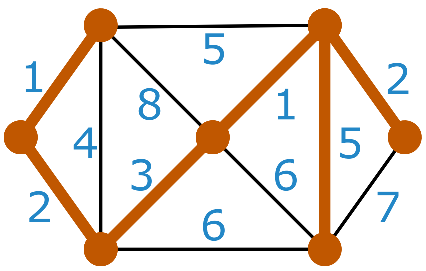
Result. Using Ideas #1 & #2, Prim’s Algorithm is $\Theta(M + N \log N)$.

Our version: $\Theta(M \log M)$.

Another greedy algorithm to find a minimum spanning tree:
Kruskal's Algorithm [J. Kruskal 1956].

Procedure

- Set edge set of tree to empty.
- Repeat:
  - Add the least-weight edge joining two vertices that cannot be reached from each other using edges added so far.
- Return edge set of tree.



Same spanning tree as Prim's Algorithm, but constructed in a different order.

To implement Kruskal's Algorithm well, we need an efficient way to check whether a vertex can be reached from another vertex.

We cover a solution to this problem soon!